Florian Cajori.

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EUCLID REVISED

NIXON

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Clarendon Press Series

EUCLID REVISED

CONTAINING

THE ESSENTIALS OF THE ELEMENTS OF PLANE GEOMETRY AS GIVEN BY EUCLID IN HIS FIRST SIX BOOKS

WITH NUMEROUS ADDITIONAL PROPOSITIONS AND EXERCISES

EDITED BY

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'GEOMETRY IN SPACE' AND 'PLANE TRIGONOMETRY WITHOUT IMAGINARIES'

THIRD EDITION

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PREFACE

TO THE SECOND EDITION

The complete *Euclid Revised* contains—

1°, Enunciations, numbered according to Euclid, of *all* the Propositions in Books i–iv and vi of his *Elements of Geometry:*

2°, Proofs of these Propositions in which the essentials of Euclid's methods are followed:

3°, an Abridgement of Book v, including so much only as is necessary to render valid the proofs of Book vi:

4°, *Addenda* at the end of each Book, arising out of its principles, in which are given—

(*a*) All the most obvious Corollaries to the Propositions;

(*b*) Some immediate Developments of the Propositions;

(*c*) The proofs of many useful Additional Theorems;

(*d*) Numerous Theorems as Exercises to be proved; accompanied by Hints towards the proofs of the more difficult:

5°, General *Addenda* arranged in Sections; wherein will be found most of the fundamental Propositions of *Maxima* and *Minima*, Concurrency and Collinearity, Centres of Similitude, Co-axal Circles, The Tangencies, Inversion, Harmonic Section, and Poles and Polars.

The whole Work is divided into two Parts.

Part I — Plane Geometry without Proportion — contains Books i–iv, with their *Addenda*.

Part II — Proportion, and Modern Geometry — contains Books v and vi, with their *Addenda;* and the eight Sections entitled *General Addenda*.

Each Part concludes with a collection of Problems for solution.

In considering what modification of Euclid's proofs might be admissible, the question at once arose—Why is it that, while in all other Sciences, text-books seldom outlive a generation, Euclid's *Elements* still hold their place as the basis of Geometry; and moreover that, in spite of the weighty arguments which have been urged against them, there can be no doubt but that a strong preponderance of feeling exists in favour of their retention in that position?

To the present Editor (after much reading about, and discussion of the question) it seems that there are *two* substantial reasons, of expediency and convenience, out of which the feeling arises.

1°, an established *order* of geometric proof is expedient for examination purposes;

2°, a recognised *numbering* of fundamental results is convenient for reference.

As co-operative reasons may be added—the fact that there is no consensus of opinion among experts that any other scheme yet proposed is superior to Euclid's; and the sentiment of repugnance at the thought of sweeping away an institution rendered venerable by the usage of more than 2000 years.

From these considerations it becomes apparent, on the one hand, that what is essential to be retained in Euclid is his order, numbering, and general mode of proof; and, on the other hand, that what is non-essential, and of small (or no) importance, is the accidental details of his proofs—whether, for example, i. 20 is proved by bisecting an angle, or producing a side.

It may strengthen this position to state the fact that there does not exist a modern edition which gives *Euclid pure and simple.*

The modifications of proof have been made solely for the sake of greater brevity, clearness, and simplicity. They are all strictly in accordance with Euclid's order and methods. In making these changes the Editor has not been guided by *à priori*

considerations of what might, could, would, or should be *thought* improvements; but by what he himself has (in many years' experience) actually found to be clearer, and to present fewer obstacles to a large number of learners of very varied age and mental calibre.

In the first edition, proofs of four Propositions in Book i, one in Book ii, and four in Book vi, were omitted; because these Propositions are neither necessary links in the chain of proof, nor of intrinsic geometrical value. Those omitted in Books i and ii are now inserted in an Appendix, to meet the requirements of examinations.

Definitions, Axioms, and Postulates, are introduced as they are needed; and certain Axioms and Postulates, tacitly assumed by Euclid, are inserted. This plan seems preferable to that of loading the beginner's mind with a string of words, many of which will not be needed till he is far advanced in the subject; and some not at all. The Index at the end gives the means of finding any one when it is wanted.

The Abridgement of Book v is given in the notation, and according to the methods set forth by the late Professor De Morgan in his *Connexion of Number and Magnitude.*

The present custom of omitting Book v, though quietly assuming such of its results as are needed in Book vi, is singularly illogical; and is indefensible on any ground, excepting that this Book has been found too difficult for the average learner. Nor does it mend the matter, but the reverse, to give—as some modern writers do—the arithmetical treatment of Proportion, which applies only to the exceptional case of commensurable magnitudes, as a substitute for a rigorous treatment applying to magnitudes of all kinds. The Editor has therefore taken special care to avoid that confusion of commensurable and incommensurable magnitudes, which arises from introducing purely

arithmetical processes in the treatment of the latter—a confusion most assuredly not to be found in Euclid. What is here given is at once strictly accurate, and quite within the capacity of any one who has capacity enough to understand Book vi.

Dominated as all teachers are by examination programmes, it may not be irrelevant to call special attention to the extraordinary anomaly prevalent in such programmes—that Book vi is usually named *without* the parts of Book v needed in its proofs. Thus the learner finds that while an iron logic is insisted on in the first four books—so that the omission of no link in the chain of proof (how simple soever) is permitted—ever after, complex principles are assumed without a hint of the incongruity. If it is necessary to prove that two sides of a triangle are greater than the third, surely it is necessary to prove *ex æquali*, *componendo*, and *alternando*. Every teacher admits the absurdity of the prevailing system; but the truth is that what does not 'pay' in examinations is not, and is not likely to be, taught.

Our appeal in this matter is not to teachers, but to Examining Boards.

But the main point which the Editor has aimed at is to give all demonstrations in their most compact form consistent with proof. His experience, as a teacher for twenty years, has shown him that NOTHING is so great a hindrance to the learner, especially when commencing *The Elements*, as Euclid's prolixity. And while to the beginner this prolixity is a stumbling-block, to the more proficient scholar it is a nuisance. A feeble learner is lost in Euclid's maze of words: while, in an examination-hall, an able candidate discards it as quite incompatible with the amount he has to get through in a limited time. In fact the *raison d'être* of this book is to give, in the clear, compact, orderly form that suits the necessities of modern examinations, some such rearrangement of Euclid, as most teachers probably find themselves compelled to

write out for successive generations of pupils. Furthermore, Euclid's text being admittedly an insufficient geometrical basis for even a very limited mathematical education, it is supplemented by additional matter sufficient for the needs of most students.

A large number of the Exercises given for solution have been frequently tested; and, besides, all of those on Books i–vi, and most of the rest, have been worked through by boys: it is therefore safe to assert that there are plenty which cannot have the charge brought against them of being 'too hard.'

As a great aid to brevity, symbols and contractions, where the symbols merely stand in place of words, are freely used; and as affording a clear, ready view of the steps of a demonstration, each step is invariably placed in a separate line: indeed throughout the book there will be observed a studious avoidance of crowding. Geometry arranged on the plan of 'herrings in a barrel' is repulsive and confusing.

In drawing the diagrams the Editor has taken pains to make them clear and accurate; and has maintained in them an exact identity of lettering with that in the corresponding text. He hopes that they will be found an attractive feature of the work.

References have been omitted because learners—

1°, very generally ignore them; and

2°, will gain greater benefit by having to hunt up the references themselves.

It is suggested that writing in the references (in pencil) should form part of the business of preparation.

Various Propositions, distinguished as A, B, C, etc., and some Corollaries, have become so stereotyped in Cambridge editions of Euclid, that they have almost got to be considered portions of his text; but they are additions made by Simson, and therefore are no more in place in what professes to give a strict list of Euclid's Propositions, than any other of the numerous additions

that have been made to his original compilation. They are therefore here relegated to the *Addenda*. Such references as Euc. i. 32, Cor. (1), or Euc. vi. C, are not true: there is nothing corresponding to them in *Euclid*. The only exception to this is that in vi. 3 the case of external bisection (sometimes called vi. A.) has been included in the proof, on the ground (noticed by many commentators) that the expression—'which also cuts the base'—indicates that the bisector was probably intended by Euclid to be either internal or external.

Special names for remarkable Points, Lines, or Theorems—particularly where such name indicates the discoverer—have been freely used. Quotations like *Ptolemy's Theorem*, *Ceva's Theorem*, *Simson's Line*, *the Pedal Triangle*, *the Orthocentre*, &c., are highly convenient and interesting. Much more historical nomenclature would have been used, but for the obscurity in which the history of geometrical invention is involved.

In putting together the *Addenda* the Editor has been mainly aided by *Lardner's Euclid*, *Thomson's Euclid*, *Catalan's Théorèmes et Problèmes de Géométrie Élémentaire*, and especially by the late Professor Townsend's *Modern Geometry:* this last work is of course *the* authority in its own department; and to it students are referred who desire further information. The *Syllabus of the Association for the Improvement of Geometrical Teaching* has also been consulted with advantage, and with regret that the plan of this book did not permit more use of it. The additions have been made solely on the ground of utility for further work; either because they give useful results or suggestive methods.

Since the appearance of the first edition the Editor has received much gratifying testimony that it has gained the approval of a large number of teachers and learners. Dissentients from its *modus operandi* there have been as a matter of course. These, however, appear to be chiefly teachers who, on *à priori* grounds,

think it not likely to suit young learners. They seem to think that there is a special educating power in much speaking, to which the brevity of this book would be fatal. This view is presumably founded on the misconception—that Euclid not only uses the syllogistic form of reasoning, but also that he gives *all* the steps of his syllogisms. Now this is so far from being the case that if one of the more elaborate Propositions (e.g. i. 47) is written out, with the full statement of every syllogistic step inserted, it will be found to extend to some three or four times the length of the original. Again, Euclid often trusts to the reader's intuition to bridge a step: e.g. in his very first Proposition the intersection of the circles is based neither on axiom or argument: it is left to intuition. Euclid omits many steps: the view taken here is that by omitting more the reasoning is more clearly apprehended—that the profuse verbiage and repetitions, which it is the aim of this volume to supersede, do not tend to greater, but to less appreciation of the logic—that the more argument is focused the clearer and stronger impression it produces—that, in fine, brevity is the soul of reasoning, as it is of wit. The Editor was brought to this conclusion solely by experience. Often has he seen a learner's (apparently hopeless) difficulties removed by the simple process of putting down in brief symbolic language, the few leading steps that constituted the essence of the proof. And with regard to the use of symbols in place of words, is not a word a symbol? Whether we write triangle, or $\triangle$, we equally use a symbol. And if the former describes the character of the figure, the latter gives a picture of it, which is better. Surely in this matter common sense must prevail.

It may perhaps be not altogether superfluous here to note that beginners in Geometry must be taught; and that merely hearing a lesson is not teaching. The young learner should have every step of every syllogism fully brought out for him by the teacher *vivâ*

voce. He should be at first taken along very slowly, with constant repetition, and his work, so to speak, prepared for him. Taught in this way the beginner soon ceases to need teaching, and can prepare his work from *Euclid Revised* with ease, accuracy, and complete grasp of its proofs.

The Editor desires here to record his thanks—

1°, to his Cambridge friends (*quondam* pupils) Professor J. Larmor, of St. John's College, Mr. R. A. H. MacFarland, of Caius College, and Mr. A. Larmor, of Clare College, for much help in revising the proof-sheets, and for many useful suggestions;

2°, to his friend, Professor Purser, of Queen's College, Belfast, for the original proofs (now for the first time printed) of the Theorems on pages 323, 350, 389, and for several original Exercises;

3°, to correspondents who have kindly pointed out mistakes in the first edition.

But it is distinctly to be understood that the Editor is alone responsible for the general plan and execution of the work: which work is an endeavour to meet what an Editorial Note in the *Messenger of Mathematics* (New Series, vol. i, p. 14) calls—"our great educational want—a reformed Euclid, as distinguished from a new Geometry."

THIRD EDITION

In this edition, besides various small changes and additions, a Section has been added on the *Modern Geometry of the Triangle* —a branch of Geometry which (to quote Mr. R. F. Davis' words at the 1888 meeting of the *A. I. G. T.*) "in its gradual development and present dimensions, is the most remarkable and interesting of recent additions to Elementary Mathematics."

The amount of matter given in this new Section is nearly that indicated in the corresponding Section of the Syllabus of the *A. I. G. T.* In its compilation the Editor was indebted to Mr. R. F. Davis for kindly reading the proof-sheets, and offering valuable suggestions thereupon; to Mr. Tucker for copies of his many papers on the subject; to Mr. Langley for MS. notes afterwards embodied in the 1888 Report of the *A. I. G. T.*; to M. Émile Vigarié for his pamphlets entitled *Géométrie du Triangle*; and especially to Mr. Milne for his liberal permission to make any use of his excellent *Companion*. As Mr. Milne's book contains (in the nine chapters due to Mr. Simmons) by far the most complete and elegantly written account, yet published, of the new Geometry, this permission was as valuable as it was generous: it has however only been sparingly used; and that mainly in the Exercises. The arrangements of proof here given are independent of any book.

The only other addition that seems to call for notice is an indication of the use of *Double Points* in the solution of some otherwise intractable Problems.

In the additional matter and Exercises the Editor has excluded all that is not Pure Geometry—the test of which he takes to be that it can be represented by diagrams. Hence the omission of any reference to Lines and Points at Infinity; and the exclusion of Exercises involving such expressions as the product of four lines.

It is gratifying to notice the much greater freedom in Geometry that has been obtained (chiefly by the action of the *A. I. G. T.*) since the publication of the First Edition of this book. It is now a fact that, in every important Examination in the United Kingdom, proofs of Euclid's Propositions other than those given by him, are admitted; though, in almost all cases, his order of Propositions is required.

The attention of teachers who are preparing pupils with a view to the *Mathematical Tripos*, is directed to the Report of the Cambridge Board of Mathematical Studies (22 February, 1890) by which it appears that the curriculum in Pure Geometry, prescribed for the first morning, is considerably extended—the extension took effect in 1893. It will be seen that, so far as the Geometry of the Point, Line and Circle is concerned, *Euclid Revised* covers exactly the ground there indicated. The companion volume (*Geometry in Space*) gives the requisite amount of pure Solid Geometry, for the same morning's paper.

According to the Editor's present intention, *Euclid Revised* now appears in its final form; and, unless for the correction of any absolute errors which may still remain undetected, will not be further altered. It now contains as much Pure Plane Geometry as is likely to be required by any Student who does not make a very exceptional speciality of that subject.

ROYAL ACADEMICAL INSTITUTION, BELFAST.

April, 1895.

CONTENTS.

Part I.

Part II.

PRELIMINARY.

Geometry is the Science which treats of the relative shape, size and position of hypothetical figures: it is based on definitions, axioms and postulates: these granted, all the rest follows by pure reasoning.

Definitions state the meanings which are to be attached to certain words.

Axioms state truths which the human mind is so constituted as to admit when the words in which they are stated are understood.

Postulates require us to admit that certain processes can be performed, or that certain statements are to be conceded.

Propositions—that is, subjects proposed for consideration—are, in geometry, of two kinds:

(1) *problems*—in which from *data* (things given) *quæsita* (things sought) are to be found, or constructed.

(2) *theorems*—in which from an *hypothesis* a specified *conclusion* is to be demonstrated.

Note—Two theorems are said to be *converse*, each of the other, when the hypothesis of each is the conclusion of the other.

Def. A **point** has position, but cannot be measured or divided.

Def. A **line** has position and length, but not breadth or thickness.

Ax. The intersections of lines are points.

Def. A line is said to be **straight** when the part of it between *any* two points in its length lies evenly between those points.

Post. Let it be granted that the idea of straightness involves, as a consequence, that two straight lines cannot have two points in common without having all intermediate points in common.

Post. Let it be granted that a straight line may be drawn from any one point to any other point.

Note—When we draw a straight line from a point A to a point B, we are said to 'join AB'; and for brevity the line terminated at A and B may be called 'the *join* of AB.'

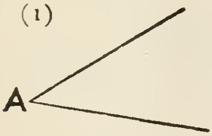
Post. Let it be granted that a straight line may be produced to any length in the same straight line.

Note—When we speak of producing a straight line AB, we are to be understood to mean that it is produced from the end B; but when we speak of producing BA, we are to be understood to mean that it is produced from the end A.

Def. When two straight lines intersect, their inclination to each other is called an **angle**; the two lines are said to *make*, or *form*, or *contain* the angle, and are called the *arms*, or *sides*, of the angle; and their point of intersection is called the *vertex* of the angle.

Note—When the straight lines forming an angle terminate at their point of intersection, as in the annexed figure (1), the angle may be denoted by a single

(1)



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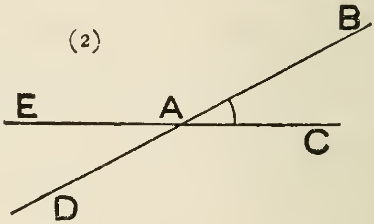


letter, placed at the common point: thus in fig. (1) the angle is denoted by $\hat{A}$.

Otherwise, as in fig. (2), there is an ambiguity in writing $\hat{A}$, for there is more than one angle formed at A; and therefore three letters are used, A being placed in the middle, and the other two defining which of the angles formed by the lines is meant: thus in fig. (2) the marked angle would be written $B\hat{A}C$.

Def. A **surface** has position, length, and breadth, but not thickness.

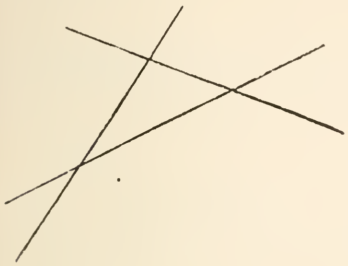
Def. The whole extent of a specified surface is called its **area.**

Def. A surface is called a **plane** when it is such that *any* two points in it being joined by a straight line, all intermediate points of the line are on the surface.

Def. A **plane figure** is a part of a plane bounded by a line or lines; and when these lines are straight it is called a **plane rectilineal figure.**

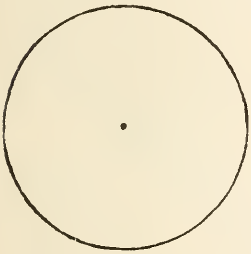
Def. If three straight lines are drawn in a plane so as to intersect two and two, the plane figure formed is called a **triangle.**

Def. The points in which lines forming a triangle intersect are called the **corners** of the triangle; the parts of the lines between the corners are called the **sides** of the triangle; and the parts of the lines not between the corners are called the **sides produced** of the triangle.



Def. The angles formed by the sides of a triangle are called the **interior angles**—or simply the **angles**—of the triangle; and the angles formed by the sides, and other sides produced are called the **exterior angles** of the triangle.

Def. A plane figure, all points of whose boundary are equally distant from a fixed point within it, is called a **circle**.

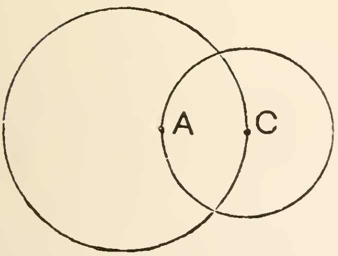


Def. The fixed point within a circle, from which all points of its boundary are equidistant, is called its **centre**.

Def. The boundary of a circle is called its **circumference.**

Def. The distance between the circumference of a circle and its centre (measured by the line joining any point in the circumference to the centre) is called its **radius.**

Note—By the nature of its definition all radii of the same circle are equal.



Ax. If the centre C of one circle is on the circumference of another circle, and a point A on the circumference of the first is within the circumference of the second, the circles will intersect in two points.

Post. Let it be granted that a circle may be described with its centre at any given point, and its circumference at a given distance from that point.

Ax. If two or more magnitudes are equal to the same magnitude they are equal to each other.

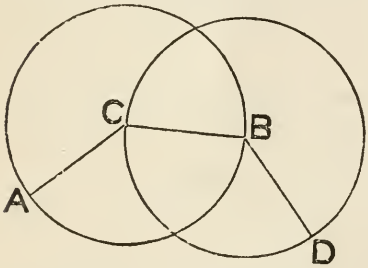
Ax. If equal magnitudes are added to other equal magnitudes (or to the same magnitude) the sums are equal.

Ax. If equal magnitudes are taken from other equal magnitudes (or from the same magnitude) greater than themselves, the remainders are equal.

Note—On the use of drawing instruments implied in the Postulates.

The Postulate on the description of a circle implies that a pair of compasses is to be used, whose points will maintain the same distance apart, as one of them is swept round the circumference, the other being fixed at the centre.

Incidentally also it assumes that compasses may be used for a limited transference of distances: for if C is the centre, and CA the distance at which the circle is to be described; then if B is another point on its circumference, the compasses, in passing round from A to B, transfers the distance CA to CB.



And again, when the points of the compasses are at B and C, if we keep B fixed, and sweep out another circle, with the point at C, then if D is any point on the circumference of the latter circle, BD, BC, and CA are all three equal to each other; for we have never changed the distance apart of the compasses' points. So that, without doing more than the Postulate demands, we have transferred the distance CA to BD.

Now compasses will preserve the distance of their points apart just as well when they are lifted, as when they are used to sweep out a circle. So that Euclid's refusal (implied in Prop. 2) to permit them to be used to transfer distances, is an arbitrary and unmeaning restriction: moreover it is a restriction never adhered to in practice. We say therefore that the use of compasses is postulated for describing circles, and for the transference of distances. Cf. 'Syllabus,' p. 1.

The Postulates on the drawing of a straight line are usually taken to mean that the use of an ungraduated straight-edge is permitted. But clearly this is not drawing a straight line in the same sense in which compasses draw a circle. The analogous mode of drawing a circle would be to make a circular disc, like a coin, and use it to trace round. And as in this case there would at once arise the question—How are we to make the disc circular? so in the other there arises the question—How are we to make the edge straight?

Curiously enough, until the year 1864 no mechanical way of drawing a straight line, similar to the mechanical way in which compasses draw a circle, was known. But in that year such a way was discovered by M. Peaucellier, a French engineer officer. The instrument he devised is known as *Peaucellier's Cell.* The learner will find it easy and interesting to make one for himself.

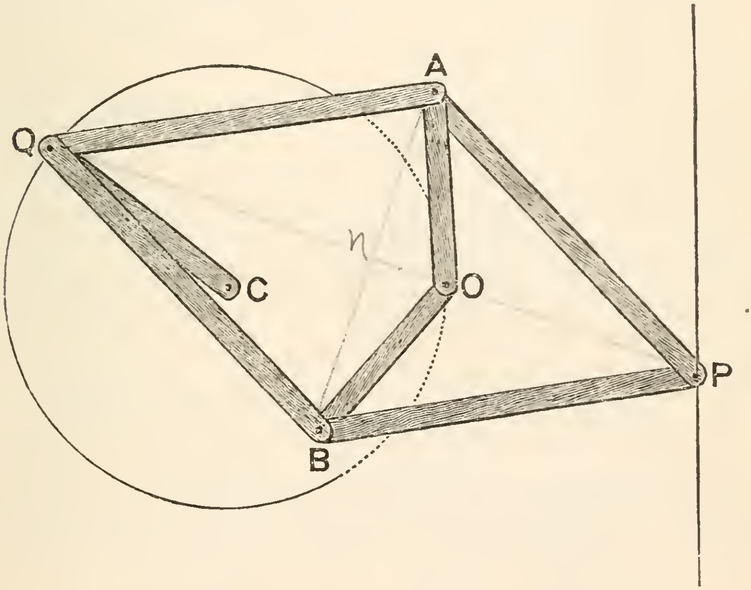
The following is an outline of its mode of construction—

Take four bars of one length, and two of another length—the two may be shorter or longer than the four; but when the two are taken shorter the instrument is more compact, and works more freely.

Suppose, in the figure, that AQ, BQ, AP, BP are the longer bars; and that AO, BO are the shorter.

Pierce holes in them at A, Q, B, P, O; and connect them through these holes by pivots, all freely moveable, except the one at O, which is to be of the nature of a nail or screw to go into a board on which the instrument is placed.

Now by means of another bar QC, of any convenient length, pivoted to the



others at Q, and to the board at C, the end Q can be made to move on the circumference of a circle. If C be so placed that CQ is equal to CO, then, as Q is moved about, P will go accurately along a straight line.

By experiment it will be found that, when CQ and CO are equal, so that P draws a straight line, Q can only be made to go a *part* of the way round the circle; but that, when CQ and CO are unequal, P describes a circle, and by properly arranging the distances, Q can be made to go entirely round one circle, and P entirely round another.

The theory of the movement will be found on p. 360.

Abbreviations.

The following symbols and contractions will be used as short-hand equivalents for the ordinary words printed in immediate connection with them.

$\therefore$ therefore	$\because$ because
$=$ is equal to	$\square^s$ parallelograms
$>$ is greater than	$\equiv$ is equal in all respects to
$<$ is less than	st. straight
$+$ together with	rt. right
$\wedge^s$ angles	pt. point
$\triangle^s$ triangles	alt. altitude
$\odot^s$ circles	sq. square
$\Vert^s$ parallels	quad. quadrilateral
$\perp^s$ perpendiculars	rem^g. remaining

And a few more similar obvious verbal contractions.

Note—The symbol $=$ is used solely as an equivalent for the words 'is equal to'; and means only that the sum total of all that is placed before the symbol is equal to the sum total of all that follows it. The symbol does not imply any equality of part to part. Nor is the symbol ever used as the equivalent of the adjective 'equal.' Thus we write 'make A equal to B'; or 'A and B are equal.' Similar remarks apply to the symbols $>$ and $<$.

The 's' after the symbols $\triangle^s$, $\Vert^s$, &c., is dropped in the singular; and the symbols $\Vert$, $\perp$, are used both for the corresponding noun and adjective.

A few other symbols are introduced on pp. 26, 53, 111, 177, 233, 243.

The period, to indicate the elision of a part of a word, as shown above, is convenient for printing; but, in *writing* a contracted word, it is best always to mark the place where the elision is made by an acute accent, in the manner following—

st′ for straight	p′t for point
rect′ for rectangle	rem′g for remaining
diag′s for diagonals	alt′s for altitudes

When one series of magnitudes is said 'to be equal to,' or 'to coincide with,' or 'to correspond to,' a second series of magnitudes, '**respectively**,' this last word indicates that such equality, or coincidence, or correspondence, is true between the magnitudes of the one series and those of the other *each to each, in the order in which they are named.*

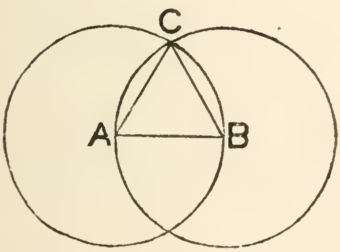
For example: if X, Y, Z are stated to be '*respectively*' equal to A, B, C, this means that $X = A$, $Y = B$, and $Z = C$.

N. B.—Until the learner is thoroughly familiar with the sequence of the main geometrical truths, as linked together by Euclid, he should write (in pencil) opposite any conclusion, depending on a previous Prop., the reference to that Prop.: e. g. on p. 12, line 14 depends on the 4th Prop. of Book i.; which should be indicated thus—

$\therefore\ \triangle DBC \equiv \triangle ACB.$ (i. 4).

Proposition 1.

PROBLEM—*On a given finite straight line to construct a triangle whose three sides shall be equal.*



Let AB be the given st. line.

With A as centre, and AB as radius, describe a ⊙; and with B as centre, and BA as radius, describe a ⊙.

Suppose C one of the pts. in which the ⊙s cut; and join CA, CB.

Since AC and AB are radii of the same ⊙,

$\therefore\ AC = AB.$

Similarly $BC = BA$:

i.e. AC and BC are each equal to the same AB.

$\therefore$ also $AC = BC$:

i.e. the three sides of the $\triangle$ ABC are equal.

Def. A triangle whose three sides are equal is said to be **equilateral.**

Proposition 2.

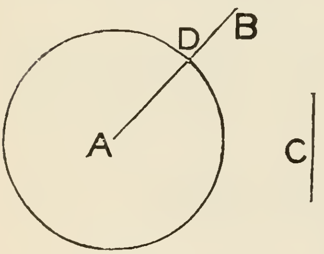
PROBLEM—*From a given point to draw a straight line equal to a given straight line.*

The simple practical solution of this Problem is to place the points of the compasses at the extremities of the given line; then to transfer them unchanged, and place one of their extremities at the given point: the join (effected by the ruler) of the given point to the point determined by the other extremity of the compasses, is the line required.

Euclid's solution will be found in the Appendix, p. 115.

Proposition 3.

PROBLEM—*From the greater of two given straight lines to cut off a part equal to the lesser.*



Let AB be the greater line, C the lesser.

With A as centre, and at a dist. which = C, describe a ⊙; and let D be the pt. where the ⊙ cuts AB.

Then AD (being a radius of the ⊙) = C:
i. e. AD is cut off as required.

Post. Let it be granted that a line, angle, or plane figure, may be conceived to be transferred, without change of magnitude, from any position to any other position.

Note—It is sometimes convenient to imagine that the transferred figure leaves its trace, or duplicate, behind it, in its old position: cf. Prop. 5.

Ax. If a plane figure, or line, or angle, can be placed on another, so that the two figures, or lines, or angles, entirely coincide, they are equal in all respects. And conversely, if two plane figures, or lines, or angles, are equal in all respects, then either one of them can be placed on the other so that the two will entirely coincide.

Def. Figures which can be made to coincide are said to be **identically equal**; and the process by which they are made to coincide is called **superposition.**

Example 1—If two straight lines are equal, then if one is placed on the other so that a pair of their extremities coincide, the other pair of their extremities must also coincide.

Example 2—If two angles are equal, then if one is placed on the other so that their vertices and a pair of their containing lines coincide, the other pair of their containing lines must also coincide.

Example 3—If two circles have equal radii, then if the centre of one is placed on the centre of the other, the circumferences of the circles will coincide.

For the radii of the circles being equal, every point on each circumference is at the same distance from the common centre, so that no point on one can be nearer to, or farther from the common centre than another.

Hence circles that have equal radii are identically equal.

Note (1)—When indicating the process of superposition, it is advisable to keep the order of the letters so as to correspond to the parts coincident: thus to say, '$\triangle$ ABC coincides with $\triangle$ XYZ,' should mean that each part of the one coincides with each part of the other, *in the order of the letters.*

Note (2)—In using the symbol $\equiv$ (called the symbol of identity) it is imperatively necessary to write the letters, indicating the parts of the two figures that are 'equal in all respects,' so that the *order* of the two sets of letters may correspond to the parts which are respectively equal: thus when we write

$$\triangle ABC \equiv \triangle XYZ,$$

it is to be understood that we imply the *set* of equalities

$$\hat{A} = \hat{X}, \quad \hat{B} = \hat{Y}, \quad \hat{C} = \hat{Z};$$

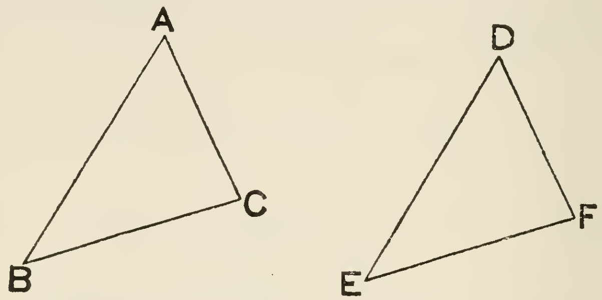
side AB = side XY, side BC = side YZ, side CA = side ZX;

and, finally, area ABC = area XYZ.

Similarly for any two figures such that one is exactly superposable on the other: for example, the symbol placed between two plane rectilineal figures of n sides each, involves 2n + 1 statements of equality.

Proposition 4.

THEOREM—*If two triangles have two sides and the included angle of the one, respectively equal to two sides and the included angle of the other, then the triangles are identically equal, and of the angles those are equal which are opposite equal sides.*



Let ABC, DEF be two $\triangle^s$, in which

$$\left.\begin{aligned} AB &= DE, \\ AC &= DF, \\ \text{and} \quad B\hat{A}C &= E\hat{D}F. \end{aligned}\right\}$$

Suppose $\triangle$ ABC to be so placed on $\triangle$ DEF that pt. A may be on pt. D, and direction of AB on direction of DE.

Then direction of AC will fall on direction of DF,

$$\because \quad B\hat{A}C = E\hat{D}F.$$

Also pt. B will coincide with pt. E,

$$\because \quad AB = DE.$$

And pt. C will coincide with pt. F,

$$\because \quad AC = DF.$$

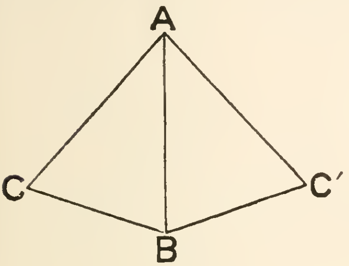
$\therefore$ also BC will coincide with EF.

So that $\triangle$ ABC can be made entirely to coincide with $\triangle$ DEF.

$$\therefore \quad \triangle ABC \equiv \triangle DEF.$$

Proposition 5.

THEOREM—*If two sides of a triangle are equal, the angles which are opposite to them are equal.*



Let ABC be a $\triangle$, in which

$$AB = AC.$$

Imagine the $\triangle$ to be turned about AB as a hinge, until it is again in the plane of the paper, so that AC takes the position AC′.

Then	AC, in the old position,			= AB in the new;	
	AB	,,	,,	= AC′	,,
and	$C\hat{A}B$	,,	,,	= $B\hat{A}C'$	,,
$\therefore$	$\triangle$ CAB	,,	,,	$\equiv \triangle$ BAC′	,,
$\therefore$	$A\hat{C}B$	,,	,,	= $A\hat{B}C'$	,,

But $A\hat{B}C'$ is only $A\hat{B}C$ turned over.

$$\therefore \quad A\hat{C}B = A\hat{B}C.$$

Def. When a triangle has two sides equal it is called **isosceles**; the third side is called **the base**; the angle opposite the base is called the **vertical angle**; and the corner of that angle is called the **vertex**.

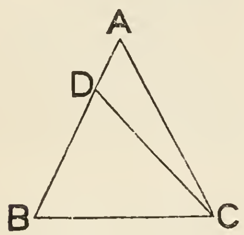
Note (1)—Prop. 5 is often enunciated thus—*The angles at the base of an isosceles triangle are equal.*

Note (2)—Euclid has a lengthy proof to show that the exterior angles, made by producing the equal sides, are equal; but he only uses the result to prove a case of Prop. 7 (which is here omitted as useless) and the result follows at once from Prop. 13, in combination with the Prop. just proved.

Euclid's proof will be found in the Appendix, p. 116.

Proposition 6.

THEOREM—*If two angles of a triangle are equal, the sides which are opposite to them are equal.*



Let ABC be a $\triangle$, in which

$$A\hat{B}C = A\hat{C}B.$$

Assume AB, AC unequal.

From the greater (BA suppose) cut off BD equal to AC; and join DC.

Then in $\triangle^s$ DBC, ACB, we have

$$\left.\begin{array}{r} DB = AC, \\ BC \text{ common}, \\ \text{and } D\hat{B}C = A\hat{C}B; \end{array}\right\}$$

$$\therefore \quad \triangle DBC \equiv \triangle ACB.$$

But this is absurd, for $\triangle$ DBC is a part of $\triangle$ ACB.

$\therefore$ the assumption that AB, AC are unequal has led to an absurdity; and $\therefore$ is not true:

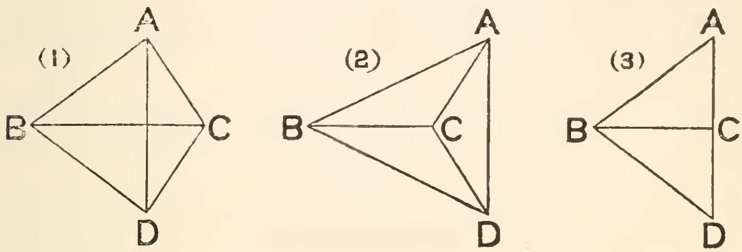
i.e. $AB = AC$.

Note (1)—Props. 5 and 6 are *converses* of each other, see p. 1.

Note (2)—The mode of proof of Prop. 6 is termed *indirect*, or *reductio ad absurdum*. It consists in assuming the contradictory of the required result, and showing that the assumption leads to an absurdity: whence, the assumption being untrue, the result is true.

Proposition 8.

THEOREM—*If two triangles have the three sides of the one respectively equal to the three sides of the other, then the triangles are identically equal, and of the angles those are equal which are opposite equal sides.*



Let the $\triangle^s$ be placed so that—

1°, a pair of equal sides may have a coincident position BC:

2°, the $\triangle^s$ may be on opposite sides of BC:

3°, the pair of sides BA, BD, terminated in B, may be an equal pair; and likewise the pair CA, CD terminated in C.

Join AD, which may pass—

across BC, fig. (1),

or outside BC, fig. (2),

or through an end C of BC, fig. (3).

Then $\because$ $BA = BD$

$\therefore$ $\widehat{BAD} = \widehat{BDA}$ in all three cases.

And $\because$ $CA = CD$,

$\therefore$ $\widehat{CAD} = \widehat{CDA}$ in figs. (1) and (2);

$\therefore$ sum of $\wedge^s$ BAD, CAD = sum of $\wedge^s$ BDA, CDA in fig. (1),
and diff. ,, ,, = diff. ,, ,, in fig. (2).

$\therefore$ in all three cases $\widehat{BAC} = \widehat{BDC}$.

So that the $\triangle^s$ come under the conditions of i. 4.

$\therefore$ $\triangle BAC \equiv \triangle BDC$.

Proposition 7.

THEOREM—*On the same base and on the same side of it there cannot be two triangles having the sides terminated at one end of the base equal, and* also *the sides terminated at the other end equal.*

This proposition is omitted because—

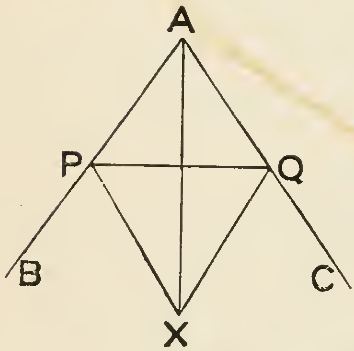
1°, it is only used by Euclid to prove Prop. 8, which has been here proved independently:

2°, it is not a theorem of any geometrical value.

Euclid's proof of it, and of Prop. 8 by means of it, will be found in the Appendix, pp. 116, 117.

Proposition 9.

PROBLEM—*To bisect a given angle.*



Let BAC be the given $\wedge$.

Take any pt. P in AB; and from AC cut off AQ, equal to AP.

On side of PQ remote from A, describe an equilat. $\triangle$ PXQ.

Join AX.

Then in $\triangle$s APX, AQX, we have

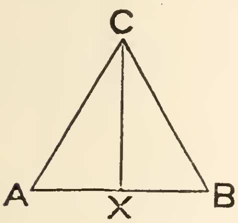
$$\left.\begin{array}{rl} & AP = AQ, \\ & AX \text{ common}, \\ \text{and} & PX = QX; \end{array}\right\}$$

$$\therefore \quad P\hat{A}X = Q\hat{A}X:$$

i.e. AX bisects $B\hat{A}C$.

Proposition 10.

PROBLEM—*To bisect a given finite straight line.*



Let AB be the given line.

On AB describe an equilat. △ ABC;

and bisect $A\hat{C}B$ by CX, meeting AB in X.

Then in △s CAX, CBX, we have

$$\left.\begin{aligned} CA &= CB, \\ CX &\text{ common}, \\ \text{and } A\hat{C}X &= B\hat{C}X; \end{aligned}\right\}$$

$$\therefore \quad AX = BX:$$

i. e. X is the mid pt. of AB.

Note—We now see the reason for describing the equilateral triangle in Prop. 9, on the side *remote* from the corner of the angle; for, in this case, if described otherwise, it would coincide with △ ABC, and the construction for CX would fail.

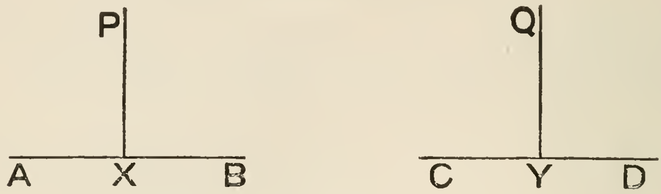
Def. When one straight line stands upon another straight line, and makes the adjacent angles equal, each of these angles is called a **right angle**; and the lines are said to be at **right angles** to each other.

Def. When two straight lines are at right angles each is called a **perpendicular** to the other.

Def. When the sum of two angles is a right angle, each is called the **complement** of the other; and the angles are said to be **complementary angles.**

Def. When the sum of two angles is two right angles, each is called the **supplement** of the other; and the angles are said to be **supplementary angles.**

Note—That all right angles are equal is easily seen by considering two sets of them to be superposed.



Let PX, QY stand on AB, CD respectively, so that

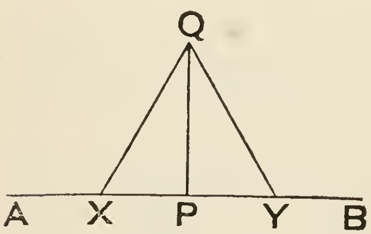
$$P\hat{X}A = \text{adjacent } P\hat{X}B,$$

$$\text{and } Q\hat{Y}C = \text{adjacent } Q\hat{Y}D.$$

Apply them so that X and Y coincide, and that AB, CD are in the same direction. Then the assumption that PX, QY are not in the same direction would obviously lead to the absurdity of an angle being both greater than, and equal to the same angle.

Proposition 11.

PROBLEM—*To draw a straight line at right angles to a given straight line from a given point in the same.*



Let AB be the given st. line; P the given point in it.

Take any pt. X in AP; and in PB take Y, so that

$$PY = PX.$$

On XY describe an equilat. △ XQY; and join QP.

Then in △s PXQ, PYQ, we have

$$\left.\begin{array}{r} PX = PY, \\ PQ \text{ common}, \\ \text{and } QX = QY; \end{array}\right\}$$

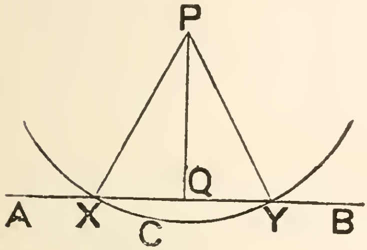
$$\therefore \quad Q\hat{P}X = Q\hat{P}Y.$$

And they are adjacent.

∴ PQ is at right angles to AB.

Proposition 12.

PROBLEM—*To draw a straight line perpendicular to a given straight line, of unlimited length, from a given point without it.*



Let AB be the given line; P the given pt. without it.

Take a pt. C on the side of AB on which P is not; and with centre P and radius PC describe a ⊙, which must cut AB in two pts., say X and Y.

Join PX, PY; and bisect $X\hat{P}Y$ by PQ, meeting AB in Q.

Then in $\triangle^s$ PQX, PQY, we have

$$\left.\begin{aligned} PX &= PY, \\ PQ &\text{ common}, \\ \text{and } X\hat{P}Q &= Y\hat{P}Q; \end{aligned}\right\}$$

$\therefore\ P\hat{Q}X = P\hat{Q}Y.$

And they are adjacent.

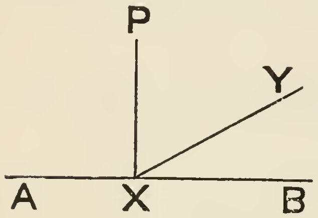
$\therefore$ PQ is $\perp$ to AB.

Def. The point in which the perpendicular to a line, from a point outside the line, meets it, is called the **foot** of the perpendicular.

Note—A line drawn, as in Prop. 11, from a point *in* a given line, so as to form two right angles with it, is said to be drawn *at right angles* to the given line; and a line drawn, as in Prop. 12, from a point *outside* a given line, so as to form two right angles with it, is said to be drawn (or dropped) *perpendicular* to the given line.

Proposition 13.

THEOREM—*The angles which one straight line makes with another, upon one side of it, are together equal to two right angles.*



Let st. line XY meet st. line AB, so as to make with AB, $A\hat{X}Y$ and $B\hat{X}Y$ on the same side of AB.

If XY is $\perp$ to AB the theorem is obvious.

But if not, draw XP $\perp$ to AB.

Then $A\hat{X}Y$ and $B\hat{X}Y$ are together made up of

$A\hat{X}P$, $P\hat{X}Y$, and $Y\hat{X}B$;

which latter three make up

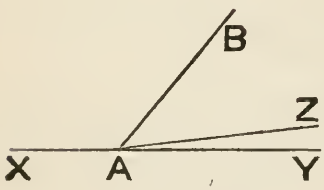
$A\hat{X}P$ and $B\hat{X}P$.

But each of these last is a right $\wedge$.

$\therefore \quad A\hat{X}Y + B\hat{X}Y =$ two rt. $\wedge^s$.

Proposition 14.

THEOREM—*If at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.*



At the point A in the st. line AB, let the st. lines AX, AY, on opposite sides of it, be so inclined that

$$B\hat{A}X + B\hat{A}Y = \text{two rt. } \wedge^s.$$

Assume that XA produced is in direction AZ.

Then, since BA meets st. line XAZ,

$$\therefore\quad B\hat{A}X + B\hat{A}Z = \text{two rt. } \wedge^s.$$

$$\therefore\quad B\hat{A}X + B\hat{A}Z = B\hat{A}X + B\hat{A}Y.$$

$\therefore$ removing $B\hat{A}X$ from each side of this equality, we have

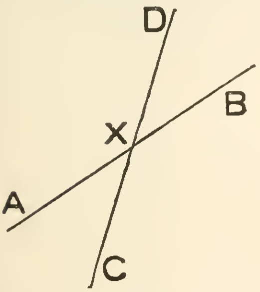
$$B\hat{A}Z = B\hat{A}Y;$$

which cannot be, unless AZ lie along AY.

$\therefore$ XA, AY are in a st. line.

Proposition 15.

THEOREM—*If two straight lines cut one another, the vertically opposite angles are equal.*



Let the two st. lines AB, CD, cut one another in X.

Then, since DX meets AXB,

$$\therefore\quad B\hat{X}D + D\hat{X}A = \text{two rt. } \wedge^s.$$

And, since AX meets DXC,

$$\therefore\quad A\hat{X}C + D\hat{X}A = \text{two rt. } \wedge^s.$$

$$\therefore\quad B\hat{X}D + D\hat{X}A = A\hat{X}C + D\hat{X}A.$$

$\therefore$, removing $D\hat{X}A$ from each side, we get

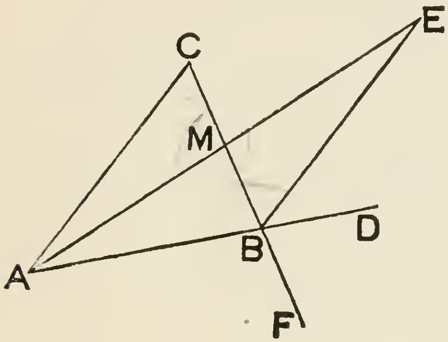
$$B\hat{X}D = A\hat{X}C.$$

Similarly it can be shown that

$$A\hat{X}D = B\hat{X}C.$$

Proposition 16.

THEOREM—*If one side of a triangle is produced, the exterior angle, so formed, is greater than either of the interior angles which are remote from it.*



Let side AB, of $\triangle$ ABC, be produced, forming the ext. $C\hat{B}D$.

Bisect CB in M.

Join AM; and produce it to E, so that ME = MA. Join EB.

Then in $\triangle^s$ BME, CMA, we have

$$\left.\begin{aligned} BM &= CM, \\ EM &= AM, \\ \text{and}\quad B\hat{M}E &= C\hat{M}A; \end{aligned}\right\}$$

$$\therefore\quad M\hat{B}E = M\hat{C}A.$$

$$\therefore\quad C\hat{B}D, \text{ which} > M\hat{B}E,$$

$$\text{also} > B\hat{C}A.$$

Similarly by producing CB to F, it could be shown that

$$A\hat{B}F \text{ (which } = C\hat{B}D) > B\hat{A}C.$$

Proposition 17.

THEOREM—*Any two angles of a triangle are together less than two right angles.*

This proposition is omitted because it is—

1°, included in i. 32:

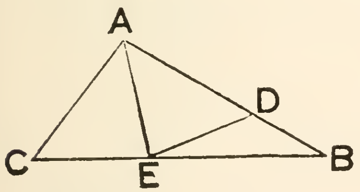
2°, not referred to in anything preceding i. 32:

3°, the converse of a theorem that Euclid states as an axiom, which will be given hereafter, (p. 31); and it is of the nature of an axiom that its converse is as axiomatic as itself.

Proof given in the Appendix, p. 118.

Proposition 18.

THEOREM—*If of two sides of a triangle one is longer than the other, then the angle which is opposite the longer side is greater than the angle which is opposite the shorter side.*



In $\triangle$ ABC suppose that

$$AB > AC.$$

Take D in AB so that

$$AD = AC.$$

Bisect $\hat{A}$ by AE, meeting BC in E. Join ED.

Then in $\triangle$s CAE, DAE, we have

$$\left.\begin{array}{r} AC = AD, \\ AE \text{ common,} \\ \text{and } C\hat{A}E = D\hat{A}E; \end{array}\right\}$$

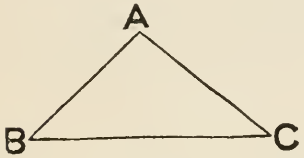
$$\therefore \quad A\hat{C}E = A\hat{D}E.$$

$$\text{But } A\hat{D}E > D\hat{B}E.$$

$$\therefore \quad A\hat{C}B > A\hat{B}C.$$

Proposition 19.

THEOREM—*If of two angles of a triangle one is greater than the other, then the side which is opposite the greater angle is longer than the side which is opposite the lesser angle.*



In $\triangle$ ABC suppose that

$$\hat{B} > \hat{C}.$$

1°, it cannot be true that AC = AB,

for this necessitates that $\hat{B} = \hat{C}$.

2°, nor can it be true that AC < AB,

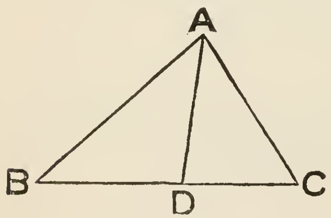
for this necessitates that $\hat{B} < \hat{C}$.

It remains $\therefore$ that AC > AB.

Note—Recollect the order of Props. 18 and 19, by noticing that they correspond to Props. 5 and 6.

Proposition 20.

THEOREM—*Any two sides of a triangle are together greater than the third side.*



Let ABC be a $\triangle$.

Take any two of its sides, AB, AC; and bisect $B\hat{A}C$ by AD, meeting BC in D.

Then $A\hat{D}B > D\hat{A}C$;

$\therefore$ also $> B\hat{A}D$.

$\therefore$ AB > BD.

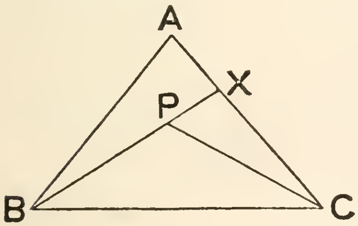
Similarly it can be shown that $AC > CD$.

$$\therefore \quad AB + AC > BC.$$

Note—This Theorem is a particular case of the more general one, that—*The shortest distance between two points is the straight line joining them.* There was a tacit assumption of the general truth as axiomatic, when every point on the circumference of a circle was defined as equidistant from the centre: such distance being measured by the straight line joining the point and centre; for clearly the word '*distance*,' in that definition, could only mean *shortest distance.*

Proposition 21.

THEOREM—*If from the ends of a side of a triangle two straight lines are drawn to a point within the triangle, the sum of them is less than the sum of the other two sides of the triangle; but they contain a greater angle.*



Let P be a pt., within the $\triangle ABC$, to which BP, CP are drawn.

Produce BP to meet AC in X.

Then $BA + AX > BX$.

$\therefore$, adding XC to each side, we get

$$BA + AC > BX + XC,$$

$$\text{i.e.} > BP + PX + XC;$$

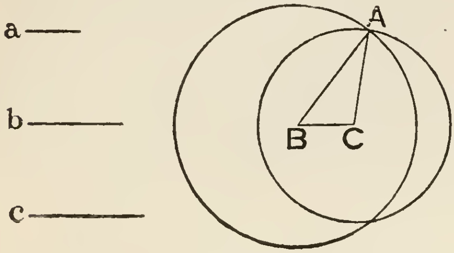
$$\therefore \quad \text{much more} > BP + PC.$$

Again $\quad B\hat{P}C > C\hat{X}P$,

$$\therefore \quad \text{much more} > B\hat{A}C.$$

Proposition 22.

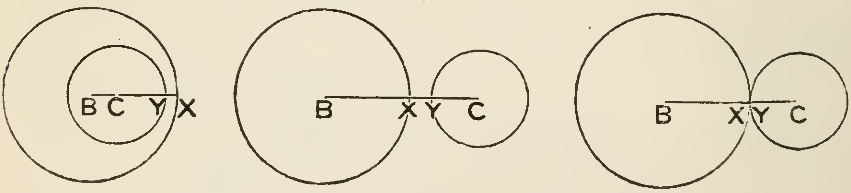
PROBLEM—*To construct a triangle when the lengths of its three sides are given.*



Let **a**, **b**, **c** be the given lengths: any two of these must together be greater than the third, or they could not be sides of a $\triangle$.

Take BC equal to **a**, and call it the base.
Suppose the side terminated in B to be **c**.
With centre B, and radius **c**, describe a $\odot$.
" C, " **b**, "
Then, as the vertex of the $\triangle$ is at a distance **c** from B,
$\therefore$ it must lie on the $\odot$ centre B.
Similarly " " C.
$\therefore$, if A is a pt. common to these $\odot$s, and AB, AC are joined,
then ABC is the reqd. $\triangle$.

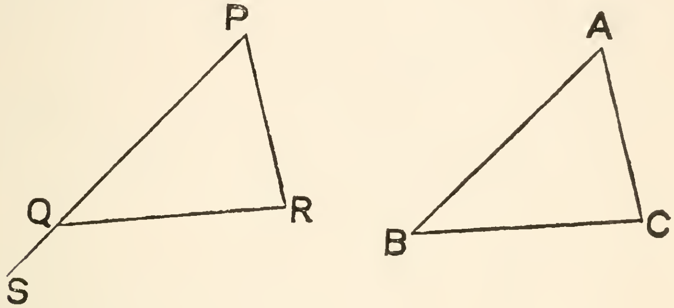
Note—That the $\odot$s must have a common pt., outside BC, may be seen by considering the only possible alternatives, diagrams of which are here drawn.



Let $\odot$s centres B, C, cut BC (or BC produced) in X, Y respectively:
1°, let $\odot$ centre C be entirely within $\odot$ centre B,
then BC + CY < BX; i.e. **a** + **b** < **c**;
2°, let $\odot$ centre C be entirely without $\odot$ centre B,
then BX + CY < BC; i.e. **c** + **b** < **a**;
3°, let $\odot$s meet on BC only, so that X, Y coincide,
then BX + CY = BC; i.e. **c** + **b** = **a**.
But if **a**, **b**, **c** can form a $\triangle$, each of these is excluded by i. 20.

Proposition 23.

PROBLEM—*At a given point, in a given straight line, to make an angle equal to a given angle.*



Let $\hat{A}$ be given; and let P be given pt. in given st. line PS.

Take any two pts. B, C in the lines which form $\hat{A}$; and join BC.

Make $\triangle$ PQR, so that one side PQ may be on PS, and equal to AB; and that QR, RP may be respectively equal to BC, CA.

Then $\triangle\ PQR \equiv \triangle\ ABC$.

$\therefore\ Q\hat{P}R = B\hat{A}C$.

i.e. $S\hat{P}R$ has been constructed as reqd.

Note—On the measurement of angles.

Angles, in common with all magnitudes, are measured by reference to some *unit*. What this unit may be is quite arbitrary; but it is usual to employ one of two. These are—

1°, the *practical unit*, which is the ninetieth part of a right angle, and is called a *degree*.

2°, the *theoretical unit*, which is that angle subtended at the centre of a circle by an arc equal in length to its radius, and is called a *radian*.

An angle has been defined as the *inclination* of two straight lines which intersect; but the idea of quantity of inclination is rather intangible. A better

way to measure the size of an angle is to consider the quantity of rotation necessary to bring one of the lines forming it into coincidence with the other; and this quantity of rotation may be best estimated by the arc intercepted between the two lines on the circumference of a circle whose centre is at the corner, or vertex of the angle.

By what has gone before we see that if a straight line is made to rotate in one plane about an extremity, until it is in line with its original direction, the amount of rotation needed to effect this indicates two right angles; so that half this amount of rotation will indicate one right angle ; and again the ninetieth part of the last will indicate one degree.

Thus, as ninety degrees will be indicated by the amount of rotation necessary to make one arm of an angle travel over the fourth part of the whole circumference of a circle, whose centre is the corner of the angle, so if the whole circumference is divided into three-hundred and sixty equal parts, each of these parts will subtend a degree at the centre.

For practical purposes the degree is subdivided into sixty equal parts, each of which is called a *minute*; and the minute is again subdivided into sixty equal parts, each of which is called a *second*. Thus, for example, the sixteenth part of a right angle contains five degrees, thirty-seven minutes, and thirty seconds; and this is indicated by the notation $5^\circ\ 37'\ 30''$.

The number of *radians* in two right angles is denoted by π.

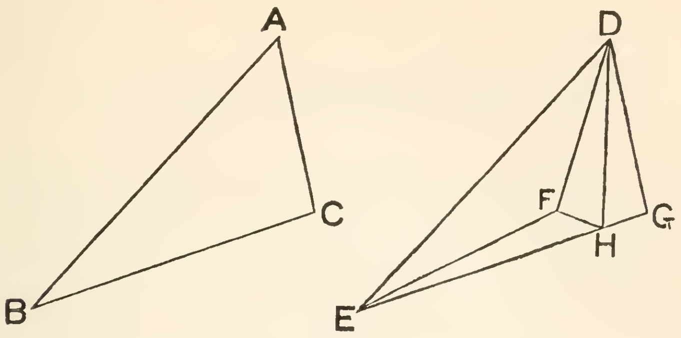
It is at once apparent that this mode of estimating angles does not limit the magnitude of an angle to less than two right angles ; and though Euclid has only given a definition of angles that will apply to angles less than two right angles, he does tacitly assume the existence of angles greater than two right angles, both in Book iii and Book vi ; and it is found, in other parts of mathematics, not only convenient but necessary to discard any such restriction.

When a plane rectilineal figure has an interior angle greater than two right angles, the figure is said to be *re-entrant*; otherwise it is said to be *convex*.

All rectilineal figures will be assumed *convex*, unless the contrary is expressly stated.

Proposition 24.

THEOREM—*If two triangles have two sides of the one equal to two sides of the other, each to each, but have the angle contained by the one pair of sides greater than the angle contained by the other pair ; then the side which is opposite the angle that is given greater, is longer than the side which is opposite the angle that is given less.*



Let ABC, DEF be two $\triangle$s, in which

$$\left.\begin{aligned} AB &= DE, \\ \text{and } AC &= DF, \\ \text{but } \widehat{BAC} &> \widehat{EDF}. \end{aligned}\right\}$$

At pt. D in DE, and on same side of DE as DF, make $\widehat{EDG}$ equal to $\widehat{BAC}$; and take G so that $DG = AC$ or DF.

Bisect $\widehat{FDG}$ by DH meeting EG in H. Join FH.

Then in $\triangle$s GHD, FHD, we have

$$\left.\begin{aligned} DG &= DF, \\ DH &\text{ common}, \\ \text{and } \widehat{GDH} &= \widehat{FDH}; \end{aligned}\right\}$$

$$\therefore \quad GH = FH.$$

Adding HE to each, we get

$$GE = FH + HE,$$

$$\therefore \quad GE > EF.$$

But in $\triangle$s ABC, DEG, since

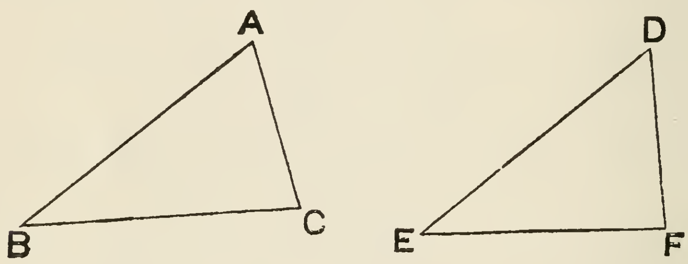
$$\left.\begin{aligned} AB &= DE, \\ AC &= DG, \\ \text{and } \widehat{BAC} &= \widehat{EDG}; \end{aligned}\right\}$$

$$\therefore \quad BC = EG.$$

$\therefore$, from above, $BC > EF$.

Proposition 25.

THEOREM—*If two triangles have two sides of the one equal to two sides of the other, each to each; but have the third side of the one longer than the third side of the other, then the angle which is opposite the side that is given longer, is greater than the angle which is opposite the side that is given shorter.*



Let ABC, DEF, be two $\triangle^s$ in which

$$\left.\begin{aligned} AB &= DE, \\ AC &= DF, \\ \text{but } BC &> EF. \end{aligned}\right\}$$

1°, it cannot be true that $\hat{A} = \hat{D}$,

for this necessitates that $BC = EF$.

2°, nor can it be true that $\hat{A} < \hat{D}$,

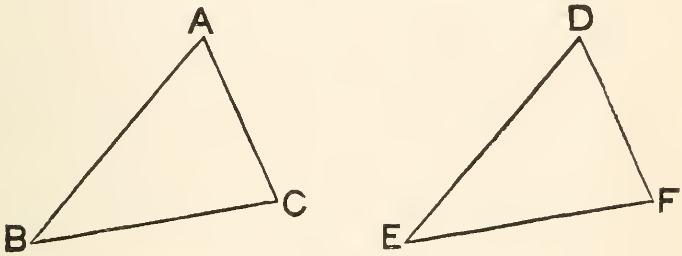
for this necessitates that $BC < EF$.

It remains $\therefore$ that $\hat{A} > \hat{D}$.

Note—Recollect the order of Props. 24 and 25, by noticing that they correspond to Props. 4 and 8.

Proposition 26. (First Part.)

THEOREM—*If two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise the two sides adjacent to these angles equal; then the triangles are identically equal, and of the sides those are equal which are opposite equal angles.*



Let ABC, DEF be two $\triangle^s$ in which

$$\left.\begin{aligned} \hat{B} &= \hat{E}, \\ \hat{C} &= \hat{F}, \\ \text{and } BC &= EF. \end{aligned}\right\}$$

Suppose $\triangle$ ABC placed on $\triangle$ DEF so that

BC coincides with EF.

Then BA will lie along ED,

$\because \quad \hat{B} = \hat{E}.$

And CA will lie along FD,

$\because \quad \hat{C} = \hat{F}.$

$\therefore$ also the intersection of BA and CA must coincide with the intersection of ED and FD:

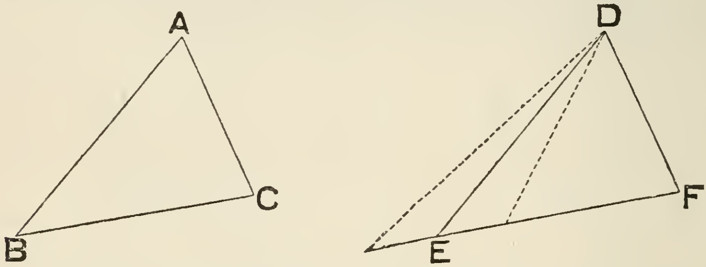
i. e. A must be on D.

So that $\triangle$ ABC can be made to coincide with $\triangle$ DEF.

$\therefore \quad \triangle ABC \equiv \triangle DEF.$

Proposition 26. (Second Part.)

THEOREM—*If two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise the sides equal which are opposite one pair of equal angles; then the triangles are identically equal, and of the sides those are equal which are opposite equal angles.*



Let ABC, DEF be two $\triangle$s in which

$$\left.\begin{aligned} \hat{B} &= \hat{E}, \\ \hat{C} &= \hat{F}, \\ \text{and}\ CA &= FD. \end{aligned}\right\}$$

Suppose $\triangle$ ABC placed on $\triangle$ DEF so that

CA coincides with FD.

Then CB will lie along FE.

$$\because\ \hat{C} = \hat{F}.$$

$\therefore$ B will be on the direction of FE.

And if AB fell anyhow excepting on DE (as in either of the positions indicated by the dotted lines) it would make with DE and EF a $\triangle$, of which the equal $\wedge$s B and E would be one exterior and the other interior and opposite.

But this cannot be.

So that $\triangle$ ABC can be made to coincide with $\triangle$ DEF.

$$\therefore\ \triangle ABC \equiv \triangle DEF.$$

If the sides AB, DE are taken equal, the proof is similar.

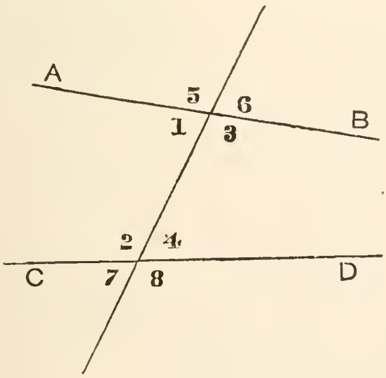
Def. If two straight lines, in the same plane, cannot, by production, be made to meet, they are said to be **parallel.**

Ax. If a straight line, meeting (or crossing) two other straight lines, makes either pair of interior angles on the same side of itself together less than two right angles, the two lines which are met (or crossed) are *not* parallel; but can be made, by production, to meet on the side on which are the two angles less than two right angles; so that the three lines will then form a triangle.

Note (1)—This is Euclid's test of parallelism; and clearly is not an axiom in the strict sense of the word axiom, given at the beginning of the book. Euclid himself considered that the converse of it (Prop. 17)—which is just as axiomatic —needed proof. He also thought it necessary to prove some theorems of a more axiomatic character, e. g. that two sides of a triangle are together greater than the third side.

Many substitutes for this axiom have been suggested: the best of them is this—"Two intersecting straight lines cannot *both* be parallel to the same straight line." (*Playfair*)

All the propositions about parallels, in which Euclid's axiom is used, can be deduced from this last axiom. It will be a good exercise for the learner to try and make the deduction for himself. In the following propositions Euclid's method is followed.



Note (2)—If two straight lines AB, CD are crossed by another straight line; and the angles thus formed are denoted by numbers, as in the figure; it is usual to call—

the pair marked 1, 4, *alternate* angles;

the pair marked 2, 3, *alternate* angles;

each of the angles marked 1, 2, 3, 4, an *interior* angle;

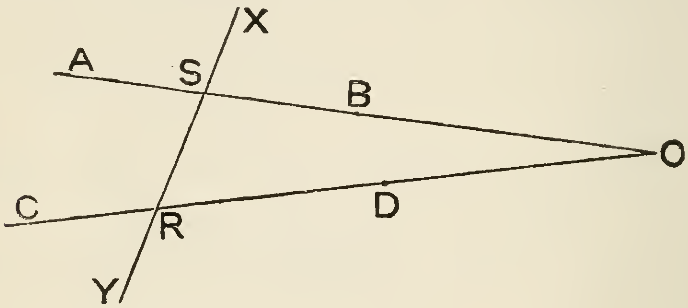
each of the angles marked 5, 6, 7, 8, an *exterior* angle;

and also each of the pairs marked 6 and 4, 5 and 2, 7 and 1, 8 and 3, *corresponding* angles.

The words *interior*, *exterior*, *alternate*, will be respectively abbreviated into *int.*, *ext.*, *altern.*

Proposition 27.

THEOREM—*If a straight line, crossing two other straight lines, makes a pair of alternate angles equal, the lines which are crossed are parallel.*



Let the st. line XY cross the st. lines AB, CD at the pts. S, R respectively; so that $\widehat{ASR}$ = altern. $\widehat{SRD}$.

If AB, CD could meet towards B and D, say in pt. O, as in the fig., then SRO would be a $\triangle$, in which

ext. $\widehat{ASR}$ = int. and opposite $\widehat{SRO}$.

But this cannot be.

Neither can they, for similar reasons, meet towards A and C.

$\therefore$ they cannot be produced to meet:

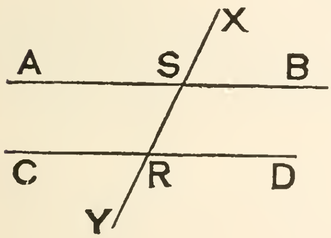
i. e. they are $\parallel$.

Proposition 28.

THEOREM—*If a straight line is drawn across two other straight lines, then the two lines which are crossed will be parallel, if—*

either (α) *an exterior angle is equal to the interior and opposite on the same side of it;*

or (β) *the two interior angles on the same side are together equal to two right angles.*



Let the st. line XY cross the st. lines AB, CD at the pts. S, R respectively.

(α) Suppose that ext. $X\hat{S}B$ = int. and opposite $S\hat{R}D$.

Then since $X\hat{S}B$ = vert. opposite $A\hat{S}R$,

$\therefore$ $A\hat{S}R$ = altern. $S\hat{R}D$.

$\therefore$ AB, CD are ∥.

(β) Suppose that $B\hat{S}R + S\hat{R}D$ = two rt. $\wedge$s.

Now $B\hat{S}R + A\hat{S}R$ = two rt. $\wedge$s.

$\therefore$ $B\hat{S}R + S\hat{R}D = B\hat{S}R + A\hat{S}R$.

$\therefore$, removing $B\hat{S}R$ from each side, we get

$S\hat{R}D$ = altern. $A\hat{S}R$.

$\therefore$ again AB, CD are ∥.

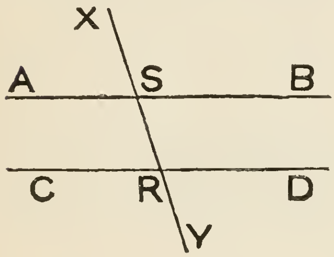
Proposition 29.

THEOREM—*If a straight line cross two parallel straight lines, it makes—*

(α) *either pair of alternate angles equal;*

(β) *each exterior angle equal to the interior and opposite angle on the same side;*

(γ) *either pair of interior angles on the same side together equal to two right angles.*



Let the st. line XY cross the $\parallel$ lines AB, CD at S, R respectively.

(α) Take a pair of altern. $\wedge^s$ ASR, SRD; and assume that one of them (say $S\hat{R}D$) $<$ the other $A\hat{S}R$.

Then, adding $B\hat{S}R$ to each, we get

$$S\hat{R}D + B\hat{S}R < A\hat{S}R + B\hat{S}R;$$

i.e. $<$ two rt. $\wedge^s$.

$\therefore$ AB, CD will meet towards B and D.

But this is contrary to the hypoth. that they are $\parallel$.

$\therefore$ the assumption is not true:

i.e. $A\hat{S}R =$ altern. $S\hat{R}D$.

Similarly $B\hat{S}R =$ altern. $S\hat{R}C$.

(β) And $X\hat{S}B$ = vert. opposite $A\hat{S}R$.

$\therefore$ $X\hat{S}B$ = int. opposite $S\hat{R}D$ on same side.

Similarly $X\hat{S}A$ = " $S\hat{R}C$ "

(γ) To each of the equals $S\hat{R}D$, $A\hat{S}R$ add $B\hat{S}R$.

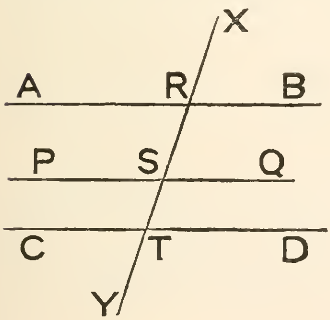
Then $S\hat{R}D + B\hat{S}R = A\hat{S}R + B\hat{S}R$,

= two rt. $\wedge^s$.

Similarly $S\hat{R}C + A\hat{S}R$ = two rt. $\wedge^s$.

Proposition 30.

THEOREM—*Straight lines which are parallel to the same straight line are parallel to each other.*



Let st. lines AB, CD be each $\parallel$ to PQ.

Draw XY across them, meeting AB, PQ, CD in R, S, T respectively.

Then $D\hat{T}S$ = altern. $T\hat{S}P$, since CD, PQ are $\parallel$,

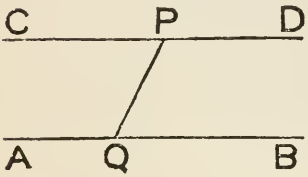
= int. opposite $S\hat{R}A$, since PQ, AB are $\parallel$:

i.e. $D\hat{T}R$ = altern. $T\hat{R}A$.

$\therefore$ CD and AB are $\parallel$.

Proposition 31.

PROBLEM—*To draw a line through a given point parallel to a given straight line.*



Let P be the given pt.; AB the given st. line.

Take any pt. Q in AB; and join PQ.

At the pt. P, in the st. line PQ, and on that side of PQ *not* the same with $\widehat{PQB}$, make $\widehat{QPC}$ equal to $\widehat{PQB}$.

Then $\because$ $\widehat{CPQ}$ = altern. $\widehat{PQB}$,

$\therefore$, if CP is produced to D, CPD is $\parallel$ to AB:

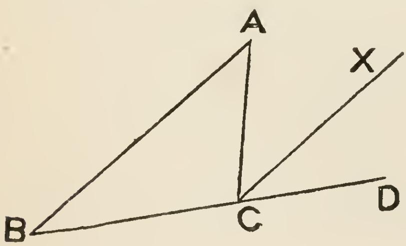
i.e. thro. P, a $\parallel$ CD has been drawn to AB.

Proposition 32.

THEOREM—*In any triangle—*

(*a*) *any exterior angle, made by producing a side, is equal to the sum of the two interior and opposite angles;*

(β) *the three interior angles are together equal to two right angles.*



Let side BC of any $\triangle$ ABC be produced to D forming the ext. $\widehat{ACD}$.

Draw CX $\parallel$ to BA.

Then $\widehat{ACX}$ = altern. $\widehat{CAB}$;

$$\text{and } D\hat{C}X = \text{int. opposite } C\hat{B}A.$$

$$\therefore \quad (\alpha)\ A\hat{C}D = C\hat{A}B + C\hat{B}A.$$

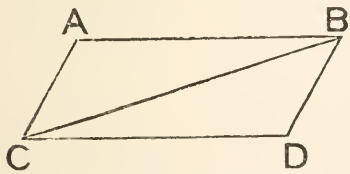
Adding $A\hat{C}B$ to each side, we get

$$(\beta)\quad C\hat{A}B + C\hat{B}A + A\hat{C}B = A\hat{C}D + A\hat{C}B,$$
$$= \text{two rt. } \wedge^{s}.$$

Def. If the extremities of two finite non-intersecting straight lines are joined, so that the joins neither cross nor are conterminous, then the lines are said to have their extremities **joined towards the same parts.**

Proposition 33.

THEOREM—*The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.*



Let AB, CD be equal and ‖ st. lines; AC, BD their joins towards the same parts.

Join BC.

Then in $\triangle^{s}$ ABC, DCB, we have

$$\left.\begin{aligned} AB &= DC, \\ CB &\text{ common,} \\ \text{and } A\hat{B}C &= \text{altern. } D\hat{C}B\,; \end{aligned}\right\}$$

$$\therefore \quad \triangle\ ABC \equiv \triangle\ DCB.$$

$$\therefore \quad AC = DB\,;$$

$$\text{and } A\hat{C}B = \text{altern. } D\hat{B}C.$$

$$\therefore \quad AC \text{ is } \parallel \text{ to } BD.$$

Def. When the opposite sides of a plane rectilineal four-sided figure are parallel, it is called a **parallelogram**; and either of the joins of its opposite corners is called a **diagonal.**

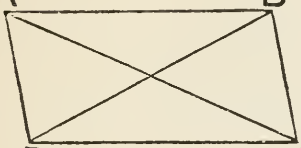
Proposition 34.

THEOREM—*In any parallelogram—*

(α) *either diagonal divides it into two identically equal triangles;*

(β) *the opposite sides are equal;*

(γ) *the opposite angles are equal.*



Let ABCD be a ▱; and AC, BD its diags.

Then in $\triangle$s ABD, CDB, we have

$$\left.\begin{array}{r} A\hat{B}D = \text{altern. } C\hat{D}B, \\ A\hat{D}B = \text{altern. } C\hat{B}D, \\ \text{and } BD \text{ common;} \end{array}\right\}$$

$\therefore \quad \triangle\, ABD \equiv \triangle\, CDB.$

Similarly $\triangle\, BAC \equiv \triangle\, DCA.$

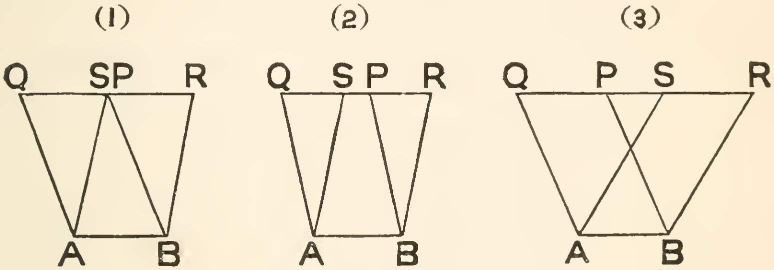
Whence also $AB = CD,\ AD = CB;$

and $B\hat{A}D = D\hat{C}B,\ A\hat{B}C = C\hat{D}A.$

Note—The *converses* of this Prop. will be found in the *Addenda*, p. 58; and also the important additions to it—*The diagonals of a parallelogram bisect each other; and conversely, a quadrilateral is a parallelogram if its diagonals bisect each other.*

Proposition 35.

THEOREM—*Parallelograms on the same base, and between the same parallels, are equal in area.*



Let ▭s ABPQ, ABRS be on same base AB,

and between same ||s AB, QR.

Then sides PQ, RS, opposite AB, may

be conterminous, fig. (1),

or overlap, fig. (2),

or be clear of each other, fig. (3).

In all three cases, since $\triangle$s ASQ, BRP, have

$$\left.\begin{aligned} S\hat{Q}A &= R\hat{P}B, \\ Q\hat{S}A &= P\hat{R}B, \\ \text{and } QA &= PB; \end{aligned}\right\}$$

$$\therefore \quad \triangle ASQ \equiv \triangle BRP.$$

But ▭ ABPQ + $\triangle$ BPR = whole fig. ABRQ,

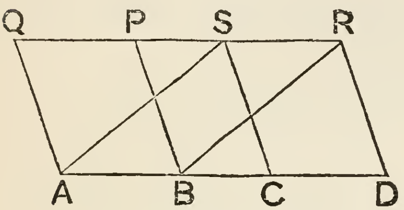
= ▭ ABRS + $\triangle$ ASQ.

$\therefore$, removing the equal $\triangle$s from each side, we have

▭ ABPQ = ▭ ABRS.

Proposition 36.

THEOREM—*Parallelograms on equal bases, and between the same parallels, are equal in area.*



Let ABPQ, CDRS be ▭s on equal bases AB, CD and between the same ||s AD, QR.

Join AS, BR.

Then since AB = CD = SR,

and AB is || to SR;

∴ AS = BR, and AS is || to BR.

∴ ABRS is a ▭.

Now ▭ ABPQ = ▭ ABRS,

∵ they are on same base, and between same ||s;

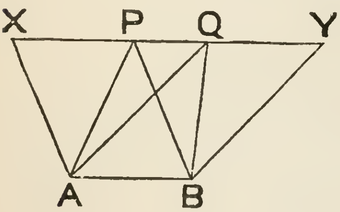
and for similar reasons,

▭ CDRS = ▭ ABRS.

∴ ▭ ABPQ = ▭ CDRS.

Proposition 37.

THEOREM—*Triangles on the same base, and between the same parallels, are equal in area.*



Let △s PAB, QAB be on same base AB, and between same ||s PQ, AB.

Draw AX || to BP, and BY || to AQ; and let them meet PQ, produced both ways, in X and Y respectively.

Then ABPX and ABYQ are ▱s.

But they are on the same base, and between same ||s.

$\therefore$ they are equal in area.

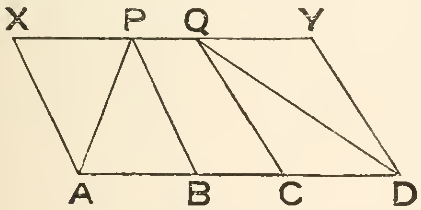
Now △ ABP = half ▱ ABPX,

and △ ABQ = half ▱ ABYQ;

$\therefore$ △ ABP = △ ABQ.

Proposition 38.

THEOREM—*Triangles upon equal bases, and between the same parallels, are equal in area.*



Let △s PAB, QCD be on equal bases AB, CD, and between the same ||s PQ, AD.

Draw AX || to BP, and DY || to CQ; and let them meet PQ, produced both ways, in X and Y respectively.

Then ABPX and CDYQ are ▱s.

But they are on equal bases, and between same ||s.

$\therefore$ they are equal in area.

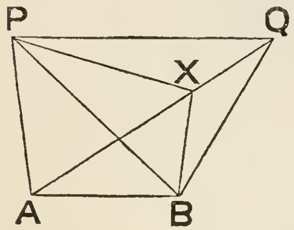
Now △ ABP = half ▱ ABPX,

and △ CDQ = half ▱ CDYQ;

$\therefore$ △ ABP = △ CDQ.

Proposition 39.

THEOREM—*Triangles of equal area, which are on the same base, and on the same side of it, are between the same parallels.*



Let $\triangle^s$ PAB, QAB be equal in area, and on the same side of same base AB.

Assume that the $\parallel$ to AB, through P, meets AQ in X; join XB.

Then $\triangle$ PAB = $\triangle$ XAB,

$\because$ they are on same base AB, and between same $\parallel^s$ PX, AB.

But $\triangle$ PAB = $\triangle$ QAB.

$\therefore$ $\triangle$ XAB = $\triangle$ QAB.

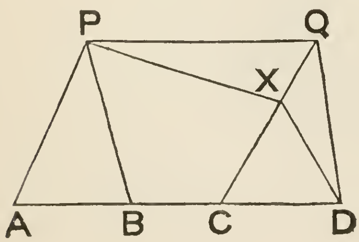
Which is impossible unless X coincide with Q.

$\therefore$ PX coincides with PQ:

i. e. PQ is $\parallel$ to AB.

Proposition 40.

THEOREM—*Triangles of equal area, which have their bases equal and in the same straight line, and which are on the same side of that line, are between the same parallels.*



Let $\triangle^s$ PAB, QCD be of equal area, on equal bases AB, CD, and on same side of st. line ABCD.

Assume that the $\parallel$ to AD through P meets CQ in X; join XD.

Then $\triangle$ PAB = $\triangle$ XCD,

$\because$ they are on equal bases AB, CD,

and between same $\parallel$s PX, AD.

But $\triangle$ PAB = $\triangle$ QCD.

$\therefore$ $\triangle$ XCD = $\triangle$ QCD.

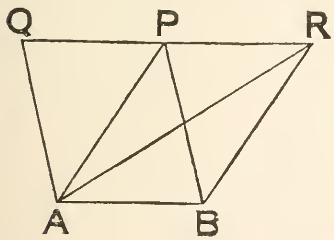
Which is impossible unless X coincide with Q.

$\therefore$ PX coincides with PQ:

i. e. PQ is $\parallel$ to ABCD.

Proposition 41.

THEOREM—*If a parallelogram and a triangle are on the same base, and between the same parallels, the parallelogram is double the triangle.*



Let ▭ ABPQ and $\triangle$ ABR be on same base AB, and between same $\parallel$s QR, AB.

Join AP.

Then $\triangle$ PAB = $\triangle$ RAB,

$\because$ they are on same base AB,

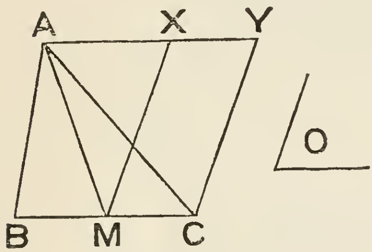
and between same $\parallel$s PR, AB.

But ▭ ABPQ is double $\triangle$ PAB.

$\therefore$ ▭ ABPQ is also double $\triangle$ RAB.

Proposition 42.

PROBLEM—*To describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let ABC be a given $\triangle$; and O a given $\wedge$.

Bisect BC in M; and join MA. Make $C\hat{M}X$ equal to $\hat{O}$; and let MX meet $\|$ to BC through A in X.

Draw CY $\|$ to MX; and let it meet AX in Y.

Then $\triangle$ AMC = $\triangle$ AMB,

$\because$ they are on equal bases, and between same $\|$s.

$\therefore$ $\triangle$ ABC is double $\triangle$ AMC.

But ▱ MY is double $\triangle$ AMC,

$\because$ they are on same base, and between same $\|$s.

$\therefore$ ▱ MY = $\triangle$ ABC;

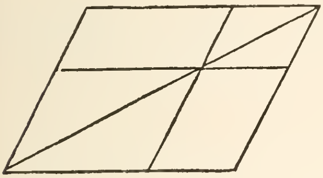
and it has $C\hat{M}X$ equal to $\hat{O}$.

i. e. MY is a ▱ described as required.

Note—This is the first step of the process by which Euclid establishes the quadrature of any rectilineal figure: that is the possibility of finding a square equal in area to a given rectilineal figure.

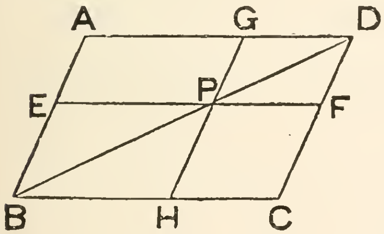
The remaining steps are given in i. 44 and 45, and ii. 14.

Def. If any point is taken on a diagonal of a parallelogram, and through it parallels drawn to the sides; then, of the four new parallelograms so formed, the two through which the diagonal passes are said to be **about the diagonal of the original parallelogram**; and the other two are called **complements** of the former two.



Proposition 43.

THEOREM—*The complements of parallelograms, which are about a diagonal of a parallelogram, are equal in area.*



Let ABCD be a ▱.

Take any pt. P in its diag. BD; and draw EPF, GPH ∥ to AD, AB respectively, forming ▱s EH, GF about the diag. BD.

Then AP, PC are the complements of EH, GF.

Since a ▱ is bisected by its diag., we have

$$\triangle\ EBP = \triangle\ HBP;$$
$$\triangle\ GPD = \triangle\ FPD;$$
$$\text{and } \triangle\ ABD = \triangle\ CBD.$$

The last line gives

$$\triangle\ EBP + \triangle\ GPD + ▱\ AP = \triangle\ HBP + \triangle\ FPD + ▱\ CP.$$

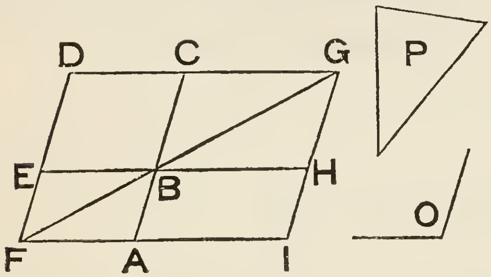
Whence, by reason of the two former lines, we get

$$▱\ AP = ▱\ CP:$$

i.e. the compts. of EH, GF are equal.

Proposition 44.

PROBLEM—*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have an angle equal to a given angle.*



Let AB be the given line; P given $\triangle$; and O given $\wedge$.

Construct a ▱ having same area as P, and an $\wedge$ equal to O; and then let it be transferred into the position BCDE, in which one of the sides BC, containing $C\hat{B}E$ which $= \hat{O}$, is in same st. line as AB.

Through A draw a $\parallel$ to BE or DC, meeting DE produced in F. Join FB.

$$\text{Then } A\hat{F}D + C\hat{D}F = \text{two rt. } \wedge^{s},$$

$$\because \quad FD \text{ meets the } \parallel^{s} AF, CD.$$

$$\therefore \quad B\hat{F}D + C\hat{D}F < \text{two rt. } \wedge^{s}.$$

$\therefore$ FB, DC will meet towards B and C, say in G.

Through G draw a line $\parallel$ to AC or FD; and let EB, FA meet this line in H, I respectively.

Then EA, CH are ▱^{s} about diag. FG of ▱ DI;
and BD, BI are complements of these.

$$\therefore \quad ▱\,BI = ▱\,BD = \triangle\,P.$$

$$\text{Also } A\hat{B}H = E\hat{B}C = \hat{O}.$$

$\therefore$ to AB has been applied a ▱ ABHI, which has same area as $\triangle$ P, and an $A\hat{B}H$ which $= \hat{O}$.

EXERCISES.

1. The bisector of the vertical angle of an isosceles triangle, 1°, bisects the base, 2°, is perpendicular to the base.

2. If two isosceles triangles are on opposite sides of the same base, the join of their vertices, 1°, bisects the base, 2°, is perpendicular to the base.

3. The bisector of the external angle, formed by producing either of the sides containing the vertical angle of an isosceles triangle, is parallel to the base.

4. If two parallelograms have an angle in one equal to an angle in the other, show that all their angles must be equal each to each.

5. If a pair of opposite sides of a parallelogram are divided into the same number of equal parts, and the corresponding points of division joined, prove that the joins will divide the parallelogram into equal parallelograms.

6. If a triangle and a parallelogram are between the same parallels, and the side of the triangle which is on one of the parallels is double the side of the parallelogram on the same parallel; show that the areas of the triangle and parallelogram are equal.

7. ABCD is a four-sided figure, of which AD is the longest side, and BC the shortest; prove that angle ABC is greater than angle ADC, and also that angle BCD is greater than angle BAD.

8. ABC is an isosceles triangle, and the bisectors of the equal angles B and C meet the opposite sides in X, Y respectively; prove that BY, CX, XY are equal.

9. ABC is any triangle, and the bisectors of the angles B and C meet in O; if the parallel to BC through O meets BA in X, and CA in Y, prove that $XY = BX + CY$.

10. Prove that any straight line through the mid point of a diagonal of a parallelogram bisects the parallelogram.

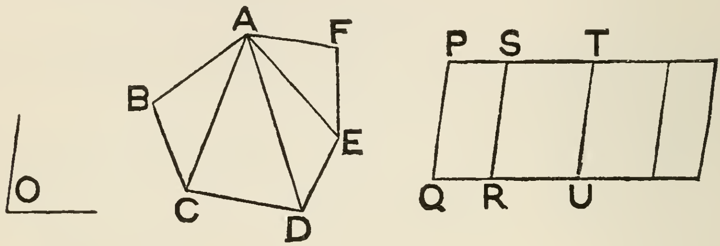
11. If the join of the extremities of two straight lines, which are equal but not parallel, makes equal angles, on the same side of itself with those lines, prove that the join of the other extremities is parallel to it.

12. If the bisector of the vertical angle of a triangle also bisects the base, prove that the triangle is isosceles.

NOTE—*Any side of a triangle may be considered as the* base, *and then the opposite angle is called the* vertical angle, *and the corner of that angle the* vertex. *The word vertex means the turning point (or top) in relation to some assigned base (or level). Hence to speak of 'the three vertices of a triangle' simultaneously is inaccurate.*

Proposition 45.

PROBLEM—*To describe a parallelogram, which shall have the same area as a given rectilineal figure, and have an angle equal to a given angle.*



Let ABCDEF be given rectilin. fig.; O given $\wedge$.

Divide given fig. into $\triangle$s by joining any one of its corners (say A) to each of the rest.

Make $\square$ PQRS, equal to $\triangle$ ABC, so that $\hat{Q} = \hat{O}$.

To RS apply $\square$ RSTU, equal to $\triangle$ ACD, so that $S\hat{R}U = \hat{O}$.

Then $S\hat{R}U = \hat{O} = P\hat{Q}R$.

$\therefore \quad S\hat{R}U + S\hat{R}Q = P\hat{Q}R + S\hat{R}Q,$

$=$ two rt. $\wedge$s, since PQ is $\parallel$ to SR.

$\therefore$ QRU is a st. line.

Again $\because$ SR meets $\parallel$s PS, QRU,

$\therefore \quad P\hat{S}R =$ altern. $S\hat{R}U$.

$\therefore \quad P\hat{S}R + T\hat{S}R = S\hat{R}U + T\hat{S}R,$

$=$ two rt. $\wedge$s, since ST is $\parallel$ to RU.

$\therefore$ PST is a st. line.

And $\because$ PQ, TU are each $\parallel$ to SR,

$\therefore$ PQ is $\parallel$ to TU.

And QU has been shown $\parallel$ to PT.

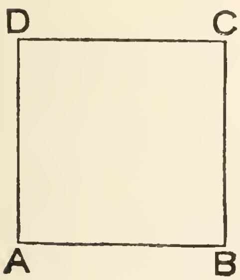
$\therefore$ PU is a $\square$; and in it $\hat{Q} = \hat{O}$.
Also $PU = PR + SU$,
$= \triangle ABC + \triangle ACD$,
$=$ fig. ABCD.

$\therefore$ PU has been constructed having same area as ABCD, and an $\wedge$ equal to $\hat{O}$.

And the process may obviously be extended to as many $\triangle$s as rectilin. fig. is divided into.

Proposition 46.

PROBLEM—*With a given straight line as one side, to describe a plane rectilineal four-sided figure, such that its sides are all equal, and its angles are all right angles.*



Let AB be the given st. line.

Draw AD $\perp$ and equal to AB.

Through B and D draw BC, DC respectively $\parallel$ to AD, AB; and let them meet in C.

Then fig. ABCD is a $\square$.
$\therefore$ $CB = AD = AB = CD$:
i. e. fig. has its sides all equal.
Also $\hat{A} + \hat{D} =$ two rt. $\wedge$s;
and $\hat{A}$ is right.
$\therefore$ $\hat{D}$ is also right.
$\therefore$ also $\hat{B}$ and $\hat{C}$, opposite these, are each right:
i. e. fig. has its $\wedge$s all right.

Def. A plane rectilineal four-sided figure, all whose sides are equal, and angles right angles, is called a **square.**

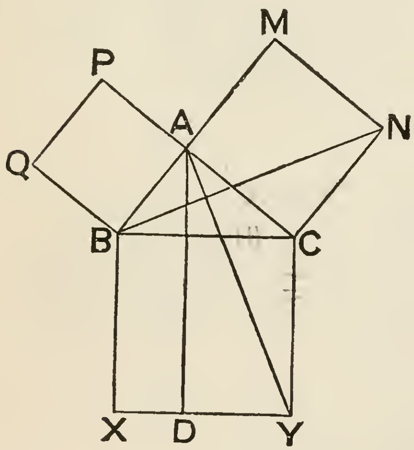
Note—From the mode of construction of a square it follows that—
1°, if straight lines are equal the squares on them are equal;
2°, if two squares are equal, any side of one is equal to any side of the other.

Def. If a triangle has one of its angles right, it is called a **right-angled triangle.**

Def. In a right-angled triangle the side opposite the right angle is called the **hypotenuse.**

Proposition 47.

THEOREM—*In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other sides.*



Let ABC be a △ having $\hat{A}$ right.

On BC describe sq. BXYC.
„ AB „ APQB.
„ AC „ AMNC.

Draw AD ‖ to BX, meeting XY in D.

Join AY, BN.

Then ∵ $B\hat{A}C$ and $C\hat{A}M$ are each right,
∴ BA and AM are in one st. line.
For a similar reason CA and AP are in one st. line.
And ∵ in △s ACY, NCB, we have
CA = CN, being sides of a sq.
CY = CB, „ „
and $A\hat{C}Y = N\hat{C}B$, for each = $A\hat{C}B$ + a rt. ∧;

$\therefore \quad \triangle ACY \equiv \triangle NCB.$

Now ▭ CD = twice △ ACY,

$\because$ they are on same base CY, and between same ||s CY, AD.

And for similar reasons

sq. CM = twice △ NCB.

$\therefore$ ▭ CD = sq. CM.

Similarly, by joining QC and AX, it could be shown that

▭ BD = sq. BP.

But ▭ CD and ▭ BD make up sq. CX.

$\therefore$ sq. on BC = sq. on AB + sq. on AC.

Def. An angle greater than a right angle is called an **obtuse angle.**

Def. If a triangle has one of its angles obtuse, it is called an **obtuse-angled triangle.**

Def. An angle less than a right angle is called an **acute angle.**

Def. If a triangle has three acute angles, it is called an **acute-angled triangle.**

Def. A plane rectilineal four-sided figure, all whose sides are equal, but its angles *not* right angles, is called a **rhombus.**

Def. A plane rectilineal four-sided figure, whose sides and angles are unrestricted, is called a **quadrilateral**; and the joins of its opposite corners are called its **diagonals.**

Def. A quadrilateral, which has *one* pair of sides parallel, is called a **trapezium** (or sometimes a **trapezoid**).

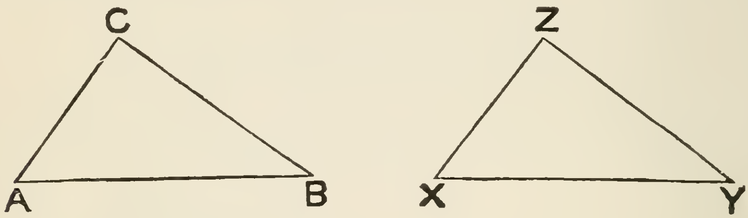
Def. A plane rectilineal figure of any number of sides is called a **polygon.**

Def. If a polygon has all its sides equal it is said to be **equilateral.**

Def. If a polygon has all its angles equal it is said to be **equiangular.**

Proposition 48.

THEOREM—*If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle contained by the latter two sides is a right angle.*



Let ABC be a $\triangle$ such that

$$\text{sq. on } AB = \text{sq. on } BC + \text{sq. on } AC.$$

Take a st. line XZ equal to AC; and draw ZY $\perp$ to it, and equal to CB.

Join XY.

$$\begin{aligned} \text{Then sq. on } XY &= \text{sq. on } XZ + \text{sq. on } ZY, \because \hat{Z} \text{ is right,} \\ &= \text{sq. on } AC + \text{sq. on } CB, \\ &= \text{sq. on } AB. \\ \therefore\quad XY &= AB. \end{aligned}$$

And since in $\triangle^s$ ACB, XZY, we have

$$\left.\begin{aligned} AC &= XZ, \\ CB &= ZY, \\ \text{and } AB &= XY; \end{aligned}\right\}$$

$$\therefore\quad A\hat{C}B = X\hat{Z}Y$$

$$\text{i.e.} = \text{a right } \angle.$$

ADDENDA TO BOOK i.

In addition to the abbreviations already indicated, the following will be used in the *Addenda* and *Exercises*.

$-$ to signify that the quantity which is placed after it is to be *subtracted* from that which goes before it.

$\sim$ to signify that the *difference* of the two quantities, between which it is placed, is to be taken: this symbol is to be used instead of the one preceding it, when we do not know which of the two quantities is the greater.

The contraction 'sq. on AB' will be still further contracted into AB^2, which is to be considered solely as an abbreviation for these words—'*the square described on the straight line* AB.'

$\ngtr$ instead of the words '*is not greater than*': this symbol includes the possibility of either $=$ or $<$ expressing the fact indicated.

Thus $A \ngtr B$ means that

$$\text{either } A = B$$
$$\text{or } A < B.$$

Similarly $\nless$ signifies '*is not less than.*'
So that $A \nless B$ means that

$$\text{either } A = B$$
$$\text{or } A > B.$$

Again $\neq$ stands for '*is not equal to.*'
Thus $A \neq B$ means that

$$\text{either } A > B$$
$$\text{or } A < B.$$

Also, for brevity, 'line' means 'straight line.'

Def. A **corollary** is an obvious inference from a demonstrated proposition.

In the *Addenda* will be found—

1°, all the most evident and important corollaries to the propositions:

2°, some useful deductions, which follow immediately from the propositions, but are not so obvious as to be properly termed corollaries:

3°, some useful theorems, depending only on Book i.

Corollaries to the Props. in Book i.

i. 5. Every equilateral triangle is also equiangular.

i. 6. Every equiangular triangle is also equilateral.

i. 13. (α) If two lines coincide in two separate points they coincide throughout their entire lengths.

(β) If two lines intersect, the sum of the four angles at their common point is equal to four right angles.

(γ) All the consecutive angles made by any number of lines drawn from one point, are together equal to four right angles.

(δ) If one line meet another, the bisectors of the supplementary angles are at right angles.

i. 16. (α) If one angle of a triangle is right, or obtuse, the other two must each be acute.

(β) Only one perpendicular can be drawn from a point outside a line to the line.

Def. Any line drawn from a point to meet a line, but *not* perpendicular to it, is called an **oblique**.

(γ) If from a point outside a line there is drawn to the line the perpendicular and any oblique, the foot of the perpendicular will lie on the acute-angled side of the oblique.

(δ) In an isosceles triangle the equal angles are acute.

i. 20. Either side of an isosceles triangle is greater than half the base.

i. 29. (α) If two intersecting lines are parallel to two others, the angle between the first pair is either equal or supplementary to the angle between the second pair.

(β) If two angles are equal, and one pair of the sides forming them are parallel, the other pair are also either parallel, or inclined at double the equal angles.

i. 32. (α) Each angle of an equilateral triangle is one-third of two right angles; or two-thirds of one right angle.

(β) If one angle of a triangle is equal to the sum of the other two it is a right angle; and conversely.

(γ) If a right-angled triangle is isosceles, each of its acute angles is half a right angle; and conversely.

(δ) If two angles of one triangle are equal to two angles of another, the remaining pair of angles are equal.

(ϵ) In a right-angled triangle the acute angles are complementary.

(ζ) In an isosceles triangle each of the two equal angles is half the supplement of the third angle, or the complement of half the third angle.

(η) The sum of the four angles of any quadrilateral is equal to four right angles.

i. 33. A quadrilateral which has two sides equal and parallel is a parallelogram.

i. 34. (α) If one angle of a parallelogram is right all its angles are right.

(β) If two adjoining sides of a parallelogram (not right-angled) are equal it is a *rhombus.*

i. 38. (α) A line from a corner of a triangle to the mid point of the opposite side bisects the triangle; and conversely.

(β) If triangles on unequal bases are between the same parallels, then the triangle on the longer base is greater than that on the shorter base; and conversely.

i. 47. (α) A square is half the square on its diagonal.

(β) In a right-angled triangle the square on one of the sides forming the right angle is equal to the difference between the squares on the hypotenuse, and on the other side.

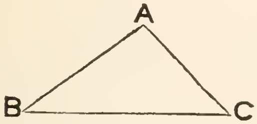
(γ) If PQ is perpendicular to AB,

$$PA^2 \sim PB^2 = QA^2 \sim QB^2.$$

THE FOLLOWING ARE SOME IMMEDIATE DEVELOPMENTS OF THE PROPS. IN BOOK i.—NOT SO OBVIOUS AS TO BE PROPERLY CALLED COROLLARIES.

Def. If a triangle has three unequal sides it is said to be **scalene**.

THEOREM (1)—*The difference between any two sides of a scalene triangle is less than the third side.*



In any $\triangle ABC$,

$$AB + AC > BC,$$

and $$AB + BC > AC.$$

If $AC < BC$, take AC from each side of the first inequality,

and then $AB > BC - AC$.

If $BC < AC$, take BC from each side of the second inequality,

and then $AB > AC - BC$.

$\therefore$ always $AB > AC \sim BC$.

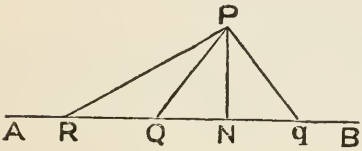
And similarly for the other pairs of sides.

THEOREM (2)—*Of all the lines that can be drawn to a given line, of unlimited length, from a given point without it—*

(α) *the perpendicular is the shortest;*

(β) *any oblique is greater than any other oblique which makes a less angle with the perpendicular;*

(γ) *to each oblique there is an equal one, on the other side of, and making the same angle with, the perpendicular.*



Let P be a pt. outside a line AB of unlimited length.

Draw to AB, PN the $\perp$, and PQ any oblique.

Then (α) $P\hat{N}Q$ (being a rt. $\wedge$) $> P\hat{Q}N$;

$\therefore \quad PN < PQ.$

Next, draw another oblique PR, so that

$$R\hat{P}N > Q\hat{P}N.$$

Then (β) since $P\hat{Q}N$ is acute,

$\therefore \quad P\hat{Q}R$ is obtuse;

and $\therefore \quad > P\hat{R}Q$:

$\therefore \quad PR > PQ.$

Lastly, draw the oblique Pq on side of PN remote from PQ, so that

$$q\hat{P}N = Q\hat{P}N.$$

Then (γ) it is clear that

$$\triangle PNq \equiv \triangle PNQ,$$

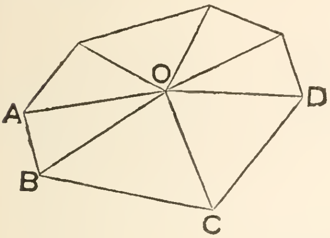
$$\therefore \quad Pq = PQ.$$

Cor. 1. There cannot be *three* equal obliques drawn from a point to a line.

Cor. 2. A circle cannot cut a line in *three* points.

Cor. 3. If from a corner of a triangle a line is drawn to the opposite side, this line is less than the greater of the sides containing the angle at the corner, if they are unequal; or than either of them, if they are equal.

THEOREM (3)—*The sum of all the interior angles of any polygon and four right angles, is equal to twice as many right angles as there are sides to the polygon.*



Let ABCD &c., be any polygon.

Take any pt. O within it; and join O to each of its corners A, B, C, D, &c.—thus forming as many $\triangle^s$ as there are sides to the pol.

Then, the three $\wedge^s$ of each $\triangle$ make up two right $\wedge^s$.
$\therefore$ sum of $\wedge^s$ of all the $\triangle^s$ = twice as many rt. $\wedge^s$ as there are sides to pol.
But all the $\wedge^s$ of the $\triangle^s$ make up the int. $\wedge^s$ of pol. + $\wedge^s$ round O.
And $\wedge^s$ round O make up four rt. $\wedge^s$.
$\therefore$ int. $\wedge^s$ of pol. + four rt. $\wedge^s$ = twice as many rt. $\wedge^s$ as there are sides to pol.

Note—If the polygon has n sides, then

all its int. $\wedge^s = (n-2)\ 180^\circ$ measured in degrees,
or $= (n-2)\ \pi$,, radians.

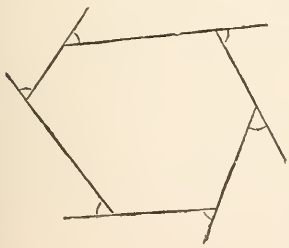
$\therefore$ if pol. is equiangular, and each of its $\wedge^s$ is a degrees, or θ radians, we have these convenient formulæ, for numerical calculation,

$$n\,a = (n-2)\ 180^\circ,$$
$$\text{or}\quad n\,\theta = (n-2)\ \pi.$$

Examples—(1) If a pol. has 20 equal $\wedge^s$, each of them $= \frac{18}{20} \times 180^\circ = 162^\circ$.

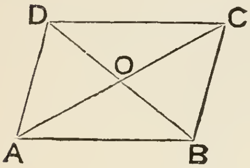
(2) Again, if each $\wedge$ of an equiangular pol. is 150°, the number of its sides $= \frac{360}{180-150} = 12$.

THEOREM (4)—*All the exterior angles of any polygon, made by producing its sides successively the same way round, together make up four right angles.*



Each ext. $\wedge$ + its adjacent int. $\wedge$
= two rt. $\wedge^s$.
$\therefore$ all ext. $\wedge^s$ + all int. $\wedge_s$
= twice as many rt. $\wedge^s$ as sides to pol.
= all int. $\wedge^s$ + four rt. $\wedge^s$.
$\therefore$ all ext. $\wedge^s$ = four rt. $\wedge^s$.

THEOREM (5)—*The diagonals of a parallelogram—*
(α) *bisect each other ;*
(β) *divide the parallelogram into four triangles of equal area.*



Let AC, BD, the diag^s^. of ▱ ABCD, cut in O.

Then in △s AOB, COD, we have

$$\left.\begin{aligned} O\hat{A}B &= O\hat{C}D, \\ O\hat{B}A &= O\hat{D}C, \\ \text{and } AB &= CD; \end{aligned}\right\}$$

$\therefore$ △ AOB ≡ △ COD,

$\therefore$ AO = CO, and BO = DO.

Also △ AOD = △ COD,

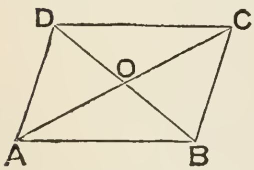
$\because$ they are on equal bases AO, CO.

Similarly △ COB = △ AOB.

$\therefore$ the areas of the 4 △s are equal.

THEOREM (6)—(Converses of i. 34, and of (α) in preceding Theorem). *A quadrilateral is a parallelogram if—*

(α) *its opposite sides are equal ;*
or (β) *its opposite angles are equal ;*
or (γ) *its diagonals bisect each other.*



Let AC, BD, the diags. of quad. ABCD, cut in O.

First, (α) $\left.\begin{aligned} &\text{if } AB = CD, \\ &\text{and } AD = BC. \end{aligned}\right\}$

Then, since in △s ABD, CDB we have

also BD common,

$\therefore$ △ ABD ≡ △ CDB.

∴ altern. ∠s made by BD with the opposite pairs of sides are equal,

∴ quad. is a ▱.

Next, (β) if $\widehat{DAB} = \widehat{DCB}$,

and $\widehat{ABC} = \widehat{ADC}$;

∴ $\widehat{DAB} + \widehat{ABC} = \widehat{DCB} + \widehat{ADC}$

= two rt. ∠s,

since the four ∠s of a quad. make up four rt. ∠s.

∴ opposite sides of quad. are ∥.

∴ it is a ▱.

$$\text{Lastly, } (\gamma) \text{ if } \left.\begin{aligned} AO &= CO, \\ \text{and } BO &= DO. \end{aligned}\right\}$$

Then, since in △s AOB, COD, we have

also $\widehat{AOB} = \widehat{COD}$, being vert^y. opposite,

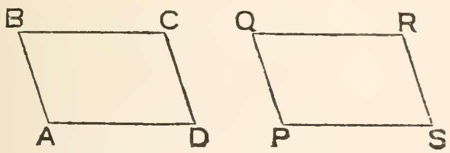
∴ △ AOB ≡ △ COD.

Similarly △ AOD ≡ △ COB.

∴ altern. ∠s made by BD with the opposite sides are equal.

∴ quad. is a ▱.

THEOREM (7)—*If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and an angle of one equal to an angle of the other, the parallelograms are identically equal.*



Let ▱s ABCD, PQRS,

$$\left.\begin{aligned} \text{have } AB &= PQ, \\ AD &= PS, \\ \text{and } \hat{A} &= \hat{P}. \end{aligned}\right\}$$

Apply ▱ AC to ▱ PR, so that

pt. A may be on pt. P, and direction of AB on that of PQ.

Then AD will be in direction of PS,

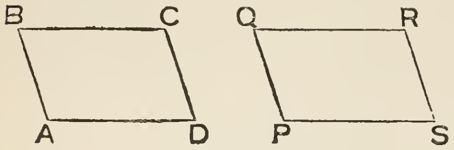
∵ $\hat{A} = \hat{P}$.

And B, D will coincide respectively with Q, S,

∵ AB = PQ, and AD = PS.

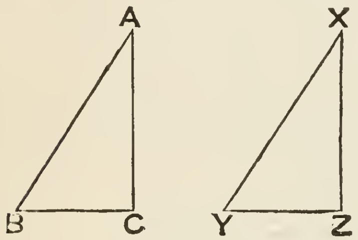
And BC will be in direction of QR,

∵ BC is ∥ to AD, and QR to PS.



Lastly DC will be in SR,
∵ DC is ∥ to AB, and SR to PQ.
∴ C, the pt. of int. of BC, DC, will coincide with R, the pt. of int. of QR, SR.
∴ ▱s will coincide entirely:
i. e. ▱ ABCD ≡ ▱ PQRS.

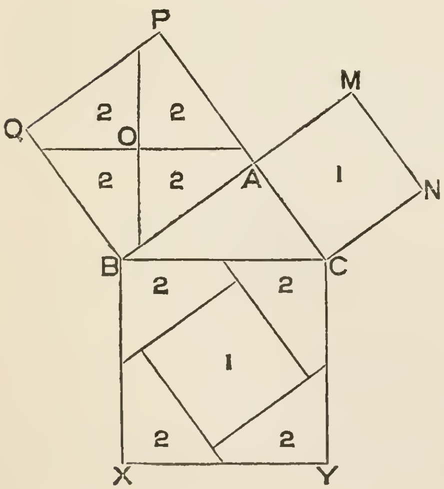
THEOREM (8)—*If two right-angled triangles have one of the sides forming the right angle in the one equal to one of the sides forming the right angle in the other, and have also their hypotenuses equal, the triangles are identically equal, and of the angles those are equal which are opposite equal sides.*



Let ABC, XYZ be △s in which

$\left.\begin{array}{l}\hat{C} \text{ and } \hat{Z} \text{ are each right,} \\ AC = XZ, \\ AB = XY.\end{array}\right\}$

Then $AC^2 + BC^2 = AB^2 = XY^2 = XZ^2 + YZ^2$.
∴ BC = YZ.
And △s come under cond^ns. of. i. 8.
∴ △ ACB ≡ △ XZY.

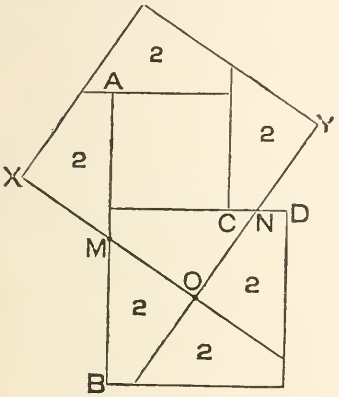


i. 47. *by dissection.*
O is centre of sq. on AB.
Through O lines are drawn ∥ and ⊥ to BC.
Through mid pts. of sides of sq. on BC lines are drawn ∥ and ⊥ to AB.
It will be found, by experimental cutting out of the pieces (and can be proved by Book i), that the pieces marked (2) are identically equal; and that the piece marked (1), in middle of sq. on BC, = sq. on AC.

Note—The above diagram should be carefully drawn on thin cardboard, and the pieces cut out, and fitted together.

This is at once the most simple, and the most ingenious of the numerous dissecting proofs that have been given of i. 47.*

The following similar dissection will cut any square into pieces that will, with any other square, form a new square.



Let AC, BD be the sqs. out of which a new sq. is to be formed.

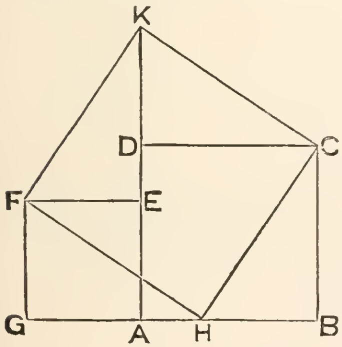
Place them so that a side of one is in same st. line with a side of the other.

Let M be mid pt. of this common line AB; N the mid pt. of CD.

Draw MO, NO through O, the centre of BD; and produce them on one way to meet sides, and the other way till MX = MO and NY = NO.

Pieces marked (2) will be found to be identically equal.*

The following is another very simple way of dissecting two squares so that the pieces will form one square.



Take any two sqs. ABCD, AEFG, and place them so that two of their sides AB, AG may be in the same st. line.

Take H, in AB, so that GH = AB.

Produce AD to K, so that EK = AB.

Join HF, HC, KF, KC.

It will be found by experiment, or it is easy to prove by Book i, that

$$\triangle CBH \equiv \triangle HGF \equiv \triangle CDK \equiv \triangle KEF;$$

and that CKFH is a square.

Thus the sqs. AC, AF have been cut into pieces which will form sq. HK.

Also these three sqs. are the sqs. described on sides of rt. angled △ CBH, so that this also proves i. 47.

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* PERIGAL—*Messenger of Mathematics, New Series*, vol. iii. p. 104.

Arithmetical Problem—To find three integer numbers such that, if they express the number of units of length of three straight lines, these lines will form a right-angled triangle.

Take any 2 nos. a and b, of which $a > b$.
Then from the identity

$$(a^2+b^2)^2 \equiv (a^2-b^2)^2 + (2ab)^2,$$

we see that $\left.\begin{array}{l} a^2 + b^2, \\ \text{and } a^2 - b^2, \\ \text{and } 2ab, \end{array}\right\}$ units of length,
will form a right-angled Δ.

Example—Take the nos. 3 and 2.

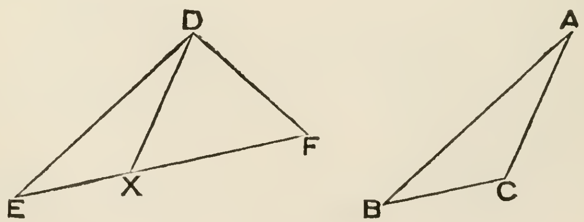
$$\begin{aligned} \text{Then} \quad 3^2 + 2^2 &= 13, \\ 3^2 - 2^2 &= 5, \\ 2 \times 3 \times 2 &= 12. \end{aligned}$$

$$\text{And } (13)^2 = 169 = 144 + 25 = (12)^2 + (5)^2.$$

So that 5, 12, 13 will represent sides of a right-angled Δ.

Here follow some useful Theorems deducible from Book i.

Theorem (9)—*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one of the equal sides in each equal, then the angles opposite to the other two equal sides are either equal or supplementary; and in the former case the triangles are identically equal.*



Let ABC, DEF be $\triangle^s$ in which

$$\left.\begin{aligned} AB &= DE, \\ AC &= DF, \\ \text{and} \quad \hat{B} &= \hat{E}. \end{aligned}\right\}$$

If $\hat{A} = E\hat{D}F$, $\triangle^s$ are identically equal by i. 4.

If not, let $\triangle$ ABC be so applied to $\triangle$ DEF that pt. B may be on pt. E, and direction of BA on ED.

Then A will coincide with D,

$$\because \quad BA = ED.$$

And BC will fall in direction of EF,

$$\because \quad \hat{B} = \hat{E}.$$

Also pt. C will be on EF (or EF produced) say at X.

$$\therefore \quad DF = AC = DX.$$

$$\therefore \quad D\hat{F}X = D\hat{X}F = \text{supp}^t.\ D\hat{X}E = \text{supp}^t.\ \hat{C}.$$

Cor. $\wedge^s$ C and F cannot be unequal and supplementary, and $\therefore$ must be equal if—

(α) they are of the same species—i. e. both right, or both acute, or both obtuse;

or (β) $\wedge^s$ B and E are right,

for then $\wedge^s$ C and F are acute;

or (γ) AC or DF $\not<$ AB or DE,

for then $\hat{B}$ or $\hat{E} \not< \hat{C}$ or $\hat{F}$,

and $\therefore$ $\wedge^s$ C and F must be acute.

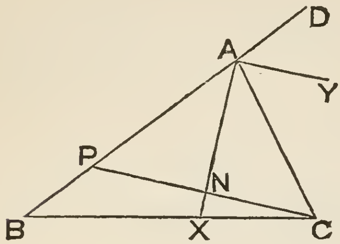
Hence when, in addition to the given cond^{ns}, we know that any one of the three (α), (β), (γ) is true, it will follow that

$$\triangle ABC \equiv \triangle DEF.$$

THEOREM (10)—*If we consider any one of the angles of a triangle as the vertical angle; then a line perpendicular to the bisector of the vertical angle makes an angle with—*

(α) *each of the lines forming the vertical angle, which is half the sum of the base angles;*

(β) *the base, which is half the difference of the base angles.*



Let ABC be a $\triangle$; AX the bisector of $\widehat{BAC}$, meeting BC in X; CNP $\perp$ to AX, meeting AB in P.

Produce BA to D; and draw AY to bisect $\widehat{CAD}$.

$$\text{Then } \widehat{XAY} \text{ is right,}$$

$$\therefore \quad AY \parallel \text{ to } CNP.$$

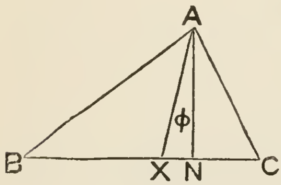
$$\therefore \quad \widehat{APC} = \widehat{DAY} = \widehat{YAC} = \widehat{ACP}.$$

But $\widehat{CAD}$ (which is double of either of these) $= \widehat{ABC} + \widehat{ACB}$.

$$\therefore \quad (\alpha) \text{ either } \widehat{APC}, \text{ or } \widehat{ACP} = \tfrac{1}{2}(\widehat{ABC} + \widehat{ACB}).$$

$$\begin{aligned} \text{And } (\beta)\ \widehat{PCB} &= \widehat{APC} - \widehat{PBC}, \\ &= \tfrac{1}{2}(\widehat{ABC} + \widehat{ACB}) - \widehat{ABC}, \\ &= \tfrac{1}{2}(\widehat{ACB} - \widehat{ABC}). \end{aligned}$$

THEOREM (11)—*If we consider any one of the angles of a triangle as the vertical angle, then the angle between the bisector of the vertical angle and the perpendicular from the vertex on the base is half the difference of the base angles.*



In $\triangle$ ABC let AX bisect $\widehat{BAC}$; and AN be $\perp$ to BC.

Denote $\widehat{XAN}$ by ϕ.

Then we have the following equalities,

$$\phi + \widehat{AXN} = \text{rt. } \wedge,$$

$$\widehat{B} + \tfrac{1}{2}\widehat{BAC} = \widehat{AXN},$$

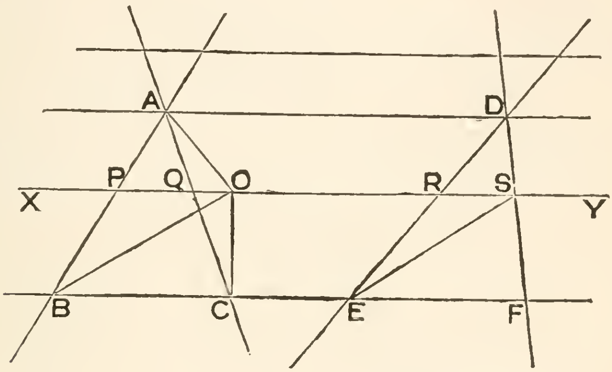
$$\phi + \text{rt. } \wedge = \widehat{AXB} = \tfrac{1}{2}\widehat{BAC} + \widehat{C}.$$

Adding corresponding sides, and omitting $\wedge^s$ common to both, we get

$$2\phi + \widehat{B} = \widehat{C},$$

$$\therefore \quad \phi = \tfrac{1}{2}(\widehat{C} - \widehat{B}).$$

THEOREM (12)—*If two triangles are on equal bases (or the same base) and between the same parallels, and any straight line is drawn parallel to the common base line, the intercepts made on it by the sides (or the sides produced) of the triangles are equal.*



Let $\triangle^s$ ABC, DEF be on equal bases BC, EF, and between same $\|^s$ AD, BF.

Let any st. line XY, $\|$ to BF, cut the sides AB, AC, DE, DF, in P, Q, R, S, respectively.

Along PQ take O, so that PO = RS.

Join BO, CO, AO, ES.

Then by sets of $\triangle^s$ on equal bases, and between same $\|^s$, we have

$\triangle$ POB = $\triangle$ RSE,

$\triangle$ APO = $\triangle$ DRS,

$\triangle$ BOC = $\triangle$ ESF.

$\therefore$, adding corresponding sides, we get

fig. ABCO = $\triangle$ DEF = $\triangle$ ABC,

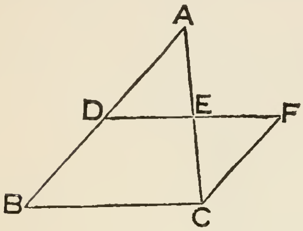
which is impossible unless O coincides with Q.

$\therefore$ PQ = RS.

If XY cuts sides produced, the proof is precisely similar.

Cor. An important particular case is that in which the triangles have a side in common, when the Theorem becomes—*The median* (p. 69) *drawn from any corner of a triangle, considered as vertex, bisects every parallel to the base, whether that parallel is terminated by the sides, or the sides produced.*

THEOREM (13)—*The join of two mid points of two sides of a triangle is parallel to the third side, and half of it.*



In $\triangle$ ABC let D be the mid pt. of AB, and E of AC.

Join DE; and produce it to F, so that EF = DE. Join FC.

Then in $\triangle^s$ AED, CEF, we have

$$\left.\begin{aligned} AE &= CE, \\ DE &= FE, \\ \text{and } \hat{AED} &= \hat{CEF}; \end{aligned}\right\}$$

$\therefore$ $\triangle$ AED $\equiv$ $\triangle$ CEF.

$\therefore$ FC = AD = BD.

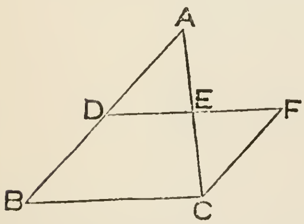
And $\hat{ADE} = \hat{CFE}$;

$\therefore$ FC is $\parallel$ to BD.

$\therefore$ FB is a $\square$.

$\therefore$ DE = $\frac{1}{2}$ DF = $\frac{1}{2}$ BC, and is $\parallel$ to BC.

THEOREM (14)—*If through the mid point of one side of a triangle a parallel is drawn to either of the other sides, it will meet the third side in its mid point.*



Thro' D, the mid pt. of AB, in $\triangle$ ABC, let DE be drawn $\parallel$ to BC, and meeting AC in E.

Draw CF, $\parallel$ to BA, to meet DE in F.

$\therefore$ BCFD is a $\square$.

$\therefore$ CF = BD = DA.

And $\hat{CFE}$ = altern. $\hat{ADE}$.

Also in $\triangle^s$ CEF, AED,

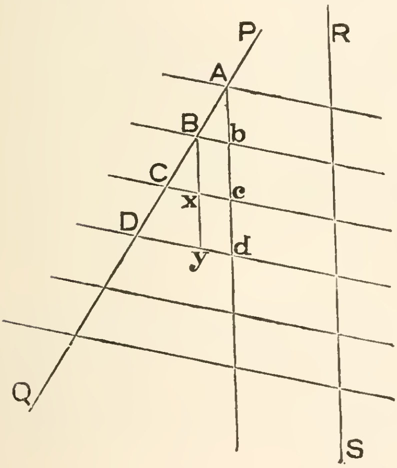
the $\wedge^s$ at E are equal.

$\therefore \quad \triangle \text{ CEF} \equiv \triangle \text{ AED}.$

$\therefore \quad \text{AE} = \text{EC}.$

Cor. By (13) $\text{DE} = \frac{1}{2}\text{ BC}.$

THEOREM (15)—*If a series of parallels intercept consecutive equal parts of any one line which they cut, they do the same on every other.*



Let a series of $\|^s$ cut the line PQ in the consecutive pts. A, B, C, D, &c. so that

$$\text{AB} = \text{BC} = \text{CD} = \text{\&c.}$$

Let RS be any other line cut by the $\|^s$.

Thro' A draw a $\|$ to RS, meeting the series of $\|^s$ in the consecutive pts. A, b, c, d, &c.

In $\triangle$ ACc, since B is mid pt. of AC, and Bb is $\|$ to Cc,

$$\therefore \quad \text{bc} = \text{Ab}.$$

Now draw Bxy $\|$ to bcd, and meeting Cc, Dd in x and y respect^y.

Then figs. Bc, xd are $\square^s$,

and x is mid pt. of By;

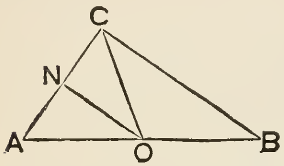
$$\therefore \quad \text{cd} = \text{xy} = \text{Bx} = \text{bc} = \text{Ab}.$$

And it is clear that the same process of proof may be extended to the rest of the consecutive intercepts on Abcd, &c.

And, by $\|^s$, each intercept on RS = the corresponding intercept on Abcd &c.

$\therefore$ the consecutive intercepts on RS are equal.

THEOREM (16)—*In a right-angled triangle the mid point of the hypotenuse is equidistant from the three corners; and conversely.*



In rt. angled $\triangle$ ACB, let O be mid pt. of hypot. AB.

Draw ON || to BC, to meet AC in N. Join CO.

Then by (14) N is mid pt. of CA.

And $\wedge^s$ at N being rt. $\wedge^s$,

$\triangle$ ONC $\equiv$ $\triangle$ ONA.

$\therefore$ OC = OA = OB.

For the converse: if in a $\triangle$ ABC, pt. O in AB is such that

OA = OB = OC;

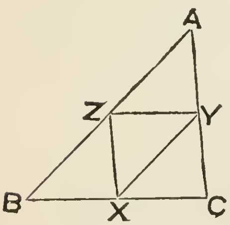
then $O\hat{A}C = O\hat{C}A$, and $O\hat{B}C = O\hat{C}B$.

$\therefore$ $A\hat{C}B = C\hat{A}B + C\hat{B}A$.

$\therefore$ $A\hat{C}B$ is rt.

Cor. If in the above AB = 2 AC, then $\triangle$ ACO is equilat.; and conversely.

THEOREM (17)—*The joins of the mid points of the sides of a triangle divide it into four triangles which are identically equal.*



In $\triangle$ ABC let X, Y, Z be the respective mid pts. of BC, CA, AB.

Then by (13) AX, BY, CZ are $\square^s$.

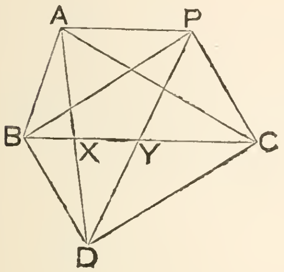
$\therefore$ their diag^s. YZ, ZX, XY divide them into identically equal $\triangle^s$:

i. e. $\triangle$ AYZ $\equiv$ $\triangle$ XZY $\equiv$ $\triangle$ ZXB $\equiv$ $\triangle$ YCX.

THEOREM (18)—*In any quadrilateral the joins of the mid points of adjacent sides form a parallelogram, whose area is half that of the quadrilateral, and whose perimeter is equal to the sum of its diagonals.*

Can be easily deduced from Theorems (13) and (17).

THEOREM (19)—*If two triangles of equal area are on opposite sides of the same base, the join of their vertices is bisected by the base; and conversely.*



Let $\triangle^s$ ABC, DBC be of equal area and on opposite sides of the same base BC.

Draw BP, CP $\|^s$ to CD, BD respect^y., and on same side of BC as $\triangle$ ABC.

Join AP.

Then BPCD is a $\square$.

$\therefore \quad \triangle$ PBC $= \triangle$ DBC $= \triangle$ ABC.

$\therefore \quad$ AP is $\|$ to BC.

Also, Y being the intersec. of diag^s. BC, PD of $\square$,

DY $=$ YP.

And since XY is $\|$ to AP,

X is mid pt. of AD.

For the converse: the same construction being made;

as before, Y is mid pt. of DP.

$\therefore$, if X is mid. pt. of AD,

then XY is $\|$ to AP.

$\therefore \quad \triangle$ ABC $= \triangle$ PBC $= \triangle$ DBC.

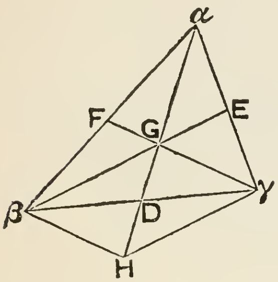
Def. Three or more straight lines which go through one point are said to be **concurrent.**

Def. Three or more points which lie on one straight line are said to be **collinear.**

Def. In a triangle the join of a corner to the mid point of the opposite side is called a **median** of the triangle.

Note—Obviously a triangle has three medians.

THEOREM (20)—*The three medians of a triangle are concurrent; and the distance of their point of concurrence from a corner is two-thirds of the length of the median along which it is measured.*



Let βE, γF be two medians of $\triangle\ \alpha\beta\gamma$, intersecting in G.

Join αG; and let it be produced to meet $\beta\gamma$ in D, and the $\parallel$ to γF thro' β in H.

Join γH.

Then F being mid pt. of $\alpha\beta$, and FG $\parallel$ to βH,

$\therefore$ G is mid pt. of αH.

Also, since G is mid of αH, and E is mid of $\alpha\gamma$,

$\therefore$ βGE is $\parallel$ to Hγ.

$\therefore$ βHγG is a $\square$.

And $\beta\gamma$, HG, its diag^{s}., bisect each other in D.

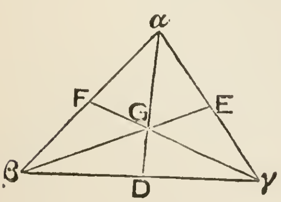
$\therefore$ the three medians are concurrent in G.

And Gα = GH = 2GD.

Def. The point of concurrence of the medians of a triangle is called its **centroid.**

Note—The centroid corresponds to the centre of mass (or gravity, which is the reason we represent it by G) of a physical triangular lamina.

THEOREM (21)—*If two sides of a triangle are unequal the median which bisects the shorter side is greater than the median which bisects the longer.*



In $\triangle\ \alpha\beta\gamma$ let D, E, F be the respective mid pts. of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

Suppose $\alpha\beta > \alpha\gamma$,

Let G be the centroid.

Then in $\triangle^s\ \alpha D\beta$, $\alpha D\gamma$, since

$$\left.\begin{aligned} \beta D &= \gamma D, \\ \alpha D &\text{ is common,} \\ \text{but } \alpha\beta &> \alpha\gamma; \end{aligned}\right\}$$

$$\therefore\quad \beta\hat{D}\alpha > \gamma\hat{D}\alpha.$$

Again in $\triangle^s\ \beta GD$, γGD, we have

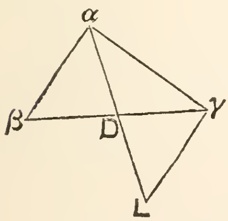
$$\left.\begin{aligned} \beta D &= \gamma D, \\ GD &\text{ common,} \\ \text{but } \beta\hat{D}G &> \gamma\hat{D}G; \end{aligned}\right\}$$

$$\therefore\quad \beta G > \gamma G,$$

$$\therefore\quad \text{by (20) } \tfrac{2}{3}\beta E > \tfrac{2}{3}\gamma F:$$

$$\text{i.e. } \beta E > \gamma F.$$

THEOREM (22)—*Of the two angles formed by a median with two adjacent sides, that which is formed with the shorter side is greater than that which is formed with the longer.*



In $\triangle\ \alpha\beta\gamma$ suppose $\alpha\beta < \alpha\gamma$.
Let D be mid pt. of $\beta\gamma$.
In αD produced take L, so that

$$DL = D\alpha.$$

Join $L\gamma$.

Then in $\triangle^s\ \alpha D\beta$, $LD\gamma$, we have

$$\left.\begin{aligned} \alpha D &= LD, \\ \beta D &= \gamma D, \\ \text{and } \alpha\hat{D}\beta &= L\hat{D}\gamma; \end{aligned}\right\}$$

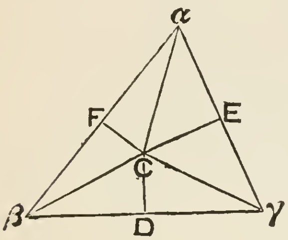
$$\therefore\quad \triangle\ \alpha D\beta \equiv \triangle\ LD\gamma.$$

$$\therefore\quad \gamma L = \alpha\beta, \text{ and } \therefore < \alpha\gamma.$$

$$\therefore\quad \gamma\hat{\alpha}D < \gamma\hat{L}D,$$

$$\therefore\quad \text{also} < \beta\hat{\alpha}D.$$

THEOREM (23)—*The three lines perpendicular to the sides of a triangle at their mid points are concurrent; and their point of concurrence is equidistant from the three corners of the triangle.*



Let D, E, F be the mid pts. of the respective sides $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ of a $\triangle\,\alpha\beta\gamma$.

Let $\perp^s$ at D and E to $\beta\gamma$, $\gamma\alpha$ meet in C. Join $C\alpha$, $C\beta$, $C\gamma$.

Then in $\triangle^s\,\beta CD$, γCD, we have

$$\left.\begin{aligned} \beta D &= \gamma D, \\ CD &\text{ common}, \\ \text{and } \beta\widehat{D}C &= \gamma\widehat{D}C; \end{aligned}\right\}$$

$\therefore\quad \beta C = \gamma C = \alpha C$, by a similar proof.

Join CF. Then in $\triangle^s\,\alpha FC$, βFC, we have

$$\left.\begin{aligned} \alpha C &= \beta C, \\ \alpha F &= \beta F, \\ \text{and } FC &\text{ common}; \end{aligned}\right\}$$

$$\therefore\quad \alpha\widehat{F}C = \beta\widehat{F}C.$$

And they are adjacent $\wedge^s$.

$\therefore\quad$ CF is $\perp$ to $\alpha\beta$:

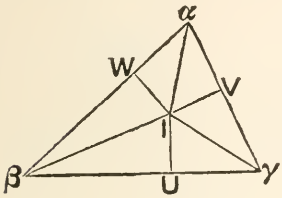
i.e. the $\perp^s$ at D, E, F are concurrent in C.

And C has been shown equidistant from α, β, γ.

Def. The point which is equidistant from the three corners of a triangle is called its **circum-centre.**

Note—The circum-centre is the centre of a circle through the three corners of the triangle; and it will be shown (iii. 10) that there is only one such circle, which is therefore called the *circumscribing circle*. Hence the name '*circum-centre*' is to be regarded solely as a contraction for the words—'*centre of the circumscribing circle.*'

THEOREM (24)—*The three lines bisecting the angles of a triangle are concurrent; and the point of their concurrence is equidistant from its sides.*



Let βI, γI bisect the $\wedge^s$ β, γ of $\triangle\, \alpha\beta\gamma$.

Draw IU, IV, IW respectively $\perp$ to $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

Then in $\triangle^s$ βIU, βIW, we have

$$\left.\begin{array}{c} \beta\hat{U}I = \beta\hat{W}I, \\ I\hat{\beta}U = I\hat{\beta}W, \\ \text{and } \beta I \text{ common;} \end{array}\right\}$$

$\therefore$ IU = IW = IV, by a similar proof.

Join Iα. Then in $\triangle^s$ αIV, αIW, we have

$$\left.\begin{array}{c} IV = IW, \\ I\alpha \text{ common,} \\ \text{and } \wedge^s \text{ at W and V right;} \end{array}\right\}$$

$\therefore$, by (8) $I\hat{\alpha}V = I\hat{\alpha}W$.

$\therefore$ the bisectors of the $\wedge^s$ of $\triangle\, \alpha\beta\gamma$ meet in I.

And I has been shown equi-distant from $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

Def. The point which is equidistant from the three sides of a triangle is called its **in-centre**.

Note—The in-centre is the centre of a circle which meets the sides of the triangle at the feet of the perpendiculars dropped from itself on the sides; and it will be shown (iii. 16) that this circle touches the sides, and it is therefore called the *inscribed circle*. As in the case of the circum-centre, the name '*in-centre*' is to be regarded solely as a contraction for the words '*centre of the inscribed circle*.' *

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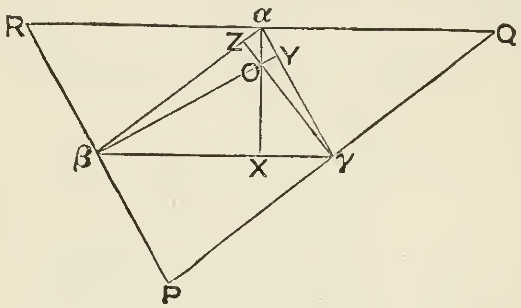
* See *Educational Times, Vol. XXXVI. New Series, No.* 269.

Def. If any one of the corners of a triangle is considered as the vertex, then the perpendicular from the vertex on its base (or the distance of the vertex from the base) is called the **altitude** of the triangle, with respect to that vertex and base.

Note—Obviously a triangle has three altitudes.

THEOREM (25)—*The three altitudes of a triangle are concurrent.*



Let $\alpha\beta\gamma$ be a Δ.

Thro' α, β, γ draw $\|^s$ to the opposite sides, forming a Δ PQR, so that P, Q, R are respect^y. opposite α, β, γ.

Then $\alpha\beta\gamma$Q and $\alpha\gamma\beta$R are $\square^s$.

$\therefore\quad \alpha Q = \beta\gamma = \alpha R.$

Similarly β and γ are mid pts. of PR, PQ.

$\therefore$, by (23), the $\perp^s$ to the sides of Δ PQR at α, β, γ, are concurrent.

Let O be their pt. of concurrence.

Then αO, produced to meet $\beta\gamma$ in X, is $\perp$ to $\beta\gamma$,

since it is $\perp$ to QR, which is $\|$ to $\beta\gamma$.

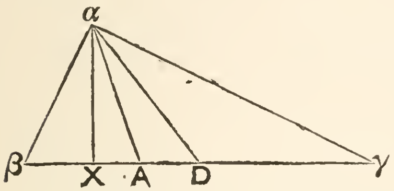
Similarly for βO and γO.

$\therefore\quad$ the three altitudes of Δ $\alpha\beta\gamma$ are concurrent in O.

Def. The point of concurrence of the three altitudes of a triangle is called its **orthocentre.**

Note—In the case of an equilateral triangle it is clear that the centroid, circum-centre, in-centre, and ortho-centre, are all coincident in one point, which may be called *the centre of the triangle.*

THEOREM (26)—*If from a vertex of a triangle, where two unequal sides meet, there is drawn the altitude, median, and bisector of the vertical angle, the last line will lie between the other two.*



In $\triangle\, \alpha\beta\gamma$, suppose that

$$\alpha\gamma > \alpha\beta.$$

Draw αX an altitude, αD a median, and αA bisecting $\widehat{\beta\alpha\gamma}$.

$$\text{Then, since } \gamma\alpha > \beta\alpha,$$

$$\therefore \text{ by (22) } \widehat{D\alpha\beta} > \widehat{D\alpha\gamma}.$$

$$\therefore \alpha A \text{ lies in } \widehat{D\alpha\beta}.$$

$$\text{Again, since } \widehat{\beta} > \widehat{\gamma},$$

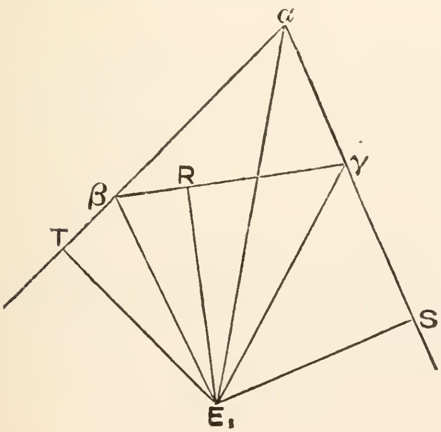
$$\therefore \widehat{X\alpha\gamma} > \widehat{X\alpha\beta}.$$

$$\therefore \alpha A \text{ lies in } \widehat{X\alpha\gamma}.$$

$$\therefore \alpha A \text{ lies in the part common to } \widehat{D\alpha\beta} \text{ and } \widehat{X\alpha\gamma}:$$

i. e. αA lies between αD and αX.

THEOREM (27)—*The bisectors of the exterior angles at two corners of a triangle are concurrent with the bisector of the interior angle at the third corner; and their point of concurrence is equidistant from the two sides produced and the third side.*



In $\triangle\ \alpha\beta\gamma$ let βE, γE bisect the ext. $\wedge^s$ at β, γ.

Draw ER, ES, ET $\perp^s$ respect^y. on sides opposite α, β, γ.

Join αE.

Then in Δ^s $ER\beta$, $ET\beta$, we have

$$\left.\begin{array}{c}\widehat{E\beta R} = \widehat{E\beta T},\\ \widehat{ER\beta} = \widehat{ET\beta},\\ \text{and } E\beta \text{ common;}\end{array}\right\}$$

$\therefore$ $ET = ER = ES$, by a similar proof.

Again, in Δ^s $E\alpha T$, $E\alpha S$, we have

$$\left.\begin{array}{l}ET = ES,\\ E\alpha \text{ common},\\ \text{also } \widehat{ET\alpha} \text{ and } \widehat{ES\alpha} \text{ each right;}\end{array}\right\}$$

$\therefore$ by (8) $\widehat{E\alpha T} = \widehat{E\alpha S}$,

$\therefore$ the bisectors of ext. $\wedge^s$ at β, γ are concurrent at E with bisector of int. $\wedge$ at α.

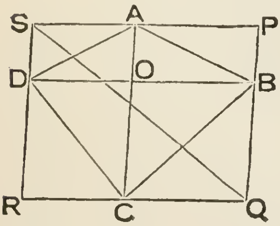
And E has been shown equidistant from βT, γS, $\beta\gamma$.

Def. The point which is equidistant from two sides produced and the third side of a triangle is called its ex-centre.

Note—The ex-centre is the centre of a circle which meets the sides of the triangle at the feet of perpendiculars dropped from itself on two produced sides, and the third side; and it will be shown (iii. 16) that this circle touches the sides, and it is therefore called the *exscribed circle* to the unproduced side that it touches. Obviously a triangle has three *ex-centres*.

As in the case of the circum-centre and in-centre, the name '*ex-centre*' is to be regarded solely as a contraction for the words '*centre of an exscribed circle*.'

THEOREM (28)—*The area of any quadrilateral is equal to that of a triangle having two sides and their included angle respectively equal to the diagonals of the quadrilateral and their included angle.*



Let ABCD be any quad.

Thro' its corners draw $\|^s$ to its diag^s, forming ▭ PQRS.

Let diag^s. cut in O.

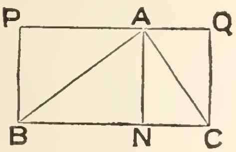
$$\text{Then } SP = \text{diag. } DOB,$$
$$QP = \text{diag. } AOC,$$
$$\text{and } \widehat{SPQ} = \widehat{AOB}.$$
$$\text{Area quad. } ABCD = \tfrac{1}{2}(PO + QO + RO + SO),$$
$$= \tfrac{1}{2} \square\, PQRS,$$
$$= \triangle\, SPQ.$$

Cor. When the diagonals are at right angles (as in the case of a rhombus)

$$\text{area} = \tfrac{1}{2} \text{ rect. under diags.}$$

Note—See beginning of Book ii. (p. 86) for definition of a **rectangle**, and meaning of the phrase rectangle *under* two lines.

THEOREM (29)—*The area of a triangle is half that of the rectangle under an altitude and its corresponding base.*



Let AN be an altitude of $\triangle$ ABC.

Draw BP, CQ $\parallel$ to AN; and PAQ $\parallel$ to BC.

$$\text{Then } \triangle\, ABC = \tfrac{1}{2} \text{ rect. } PBCQ,$$
$$= \tfrac{1}{2} \text{ rect. under } AN, BC.$$

And similarly for either of the other altitudes.

Cor. (1). The area of any quadrilateral is half the rectangle under one of its diagonals and the sum of the perpendiculars on that diagonal from the corners through which it does not pass.

Cor. (2). If r is the distance of the in-centre from each of the sides of a triangle ABC, then

$$\text{area } \triangle\, ABC = \tfrac{1}{2} r\,(AB + BC + CA).$$

Cor. (3). The area of a trapezium is half the rectangle under the sum of its parallel sides and the distance between them.

Cor. (4). Of the three quantities *area, base, altitude,* if two $\triangle^s$ have any two of the one $\triangle$ respectively equal to the corresponding two of the other $\triangle$, then the remaining ones are equal.

EXERCISES ON BOOK I.

The Corollaries (pp. 54, 55) will afford very easy Exercises for Beginners.

NOTE—*The following Exercises are all Theorems to be proved.*

1. In Prop. 1, if AB produced meets a circle in X, and D is the second point of intersection of the circles; prove that triangle CXD is equilateral.

2. If ABC is an equilateral triangle, and BC is produced to D, so that CD, CB are equal; prove that DA is perpendicular to AB.

3. Show that the last exercise gives a means of drawing a perpendicular to a terminated straight line, at an extremity, without producing it.

4. The diagonals of a right-angled parallelogram are equal.

5. If a quadrilateral is bisected by each of its diagonals, it is a parallelogram.

6. Triangles on equal bases and of equal altitudes, are of equal area.

7. If a line is terminated by two parallels, all lines through its mid point, terminated by the parallels, are bisected at that point.

8. If two sides of a triangle are unequal, any line drawn from their point of intersection to meet the third side, is less than the greater of them.

9. If the diagonals of a quadrilateral are at right angles, the sum of the squares on a pair of its opposite sides is equal to the sum of the squares on the other pair.

10. If two lines intersecting at X are respectively perpendicular to two lines intersecting at Y, each angle at X is either equal or supplementary to each angle at Y.

11. If any point is taken in the bisector of an angle, it is equally distant from the lines forming the angle.

12. If from any point in the bisector of an angle, parallels are drawn to the lines forming the angle, so as to be terminated by these lines, the parallels so drawn are equal.

13. If in triangle ABC, AX, bisecting angle BAC, cuts BC in X, and BX is greater than CX, then AB is greater than AC; and conversely.

14. If an altitude of a triangle divides its base unequally, the greater segment is adjacent to the greater side.

15. The bisectors of a pair of opposite angles of a parallelogram are either parallel, or coincident. (Cf. Ex. 94.)

16. If on the sides of a triangle ABC, equilateral triangles BCD, CAE, ABF are drawn, all external to ABC; then AD, BE, CF are equal.

17. If in a triangle ABC, AD is drawn perpendicular to BC; then—

$$AB^2 + CD^2 = AC^2 + BD^2.$$

18. If points are taken in each side of an equilateral triangle, so as to be equidistant from the corners, the distances being measured the same way round the triangle, their joins will form an equilateral triangle.

19. A theorem similar to the last is true when a square is substituted for an equilateral triangle.

20. If the alternate sides of a polygon of n sides (n being greater than 4) are produced to meet, forming a star-shaped figure, the sum of all the angles at the star points $= (n-4)\,\pi$.

NOTE—*Deduce from Addenda* (3) *and* (4).

21. The lines which bisect the angles of a parallelogram, form a right-angled parallelogram, whose diagonals are parallel to the sides of the original parallelogram.

22. If two opposite sides of a parallelogram are bisected, the lines drawn from the points of bisection to the opposite corners will trisect a diagonal.

23. If any point in a diagonal of a parallelogram is joined to its corners, the parallelogram is divided into two pairs of equal triangles.

24. If of the four triangles into which its diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has a pair of opposite sides parallel.

25. The angle between the bisectors of two external angles of a triangle is equal to half the sum of the adjacent internal angles.

26. If a quadrilateral is bisected by a diagonal, this diagonal bisects the other diagonal.

NOTE—*See Addenda* (19).

27. If A, B are points in one, and C, D points in another of two parallels; and AD, BC cut in E; then area AEC is equal to area BED.

28. If P is a point in side AB of parallelogram ABCD, and PC, PD are joined; then

$$\triangle PAD + \triangle PBC = \triangle PDC.$$

29. In any triangle ABC, if BP, CQ are perpendiculars on any line through A; and M is the mid point of BC; then MP is equal to MQ.

NOTE—*Draw* MX *perpendicular to* PQ; *and use Addenda* (15).

30. If the side BC of a triangle ABC is produced to D, and AE is drawn bisecting angle BAC, and meeting BC in E; then twice the angle AED is equal to the sum of the angles ABD and ACD.

31. An angle of a triangle is right, acute, or obtuse, according as the median drawn from its corner is equal to, greater than, or less than half the side it bisects.

32. If AD is parallel to BC ; AC, BD meet in E ; and BC is produced to P, so that area PEB is equal to area ABC ; then PD is parallel to AC.

33. If from any two points of a fixed line, lines drawn to each of two fixed points are equal, the join of the fixed points is perpendicular to the fixed line.

34. If a line is drawn from a given point, to make a given angle, with a given line ; its length, and the points of its intersection with the given line are known.

35. The perimeter of an isosceles triangle is greater than the perimeter of the rectangle of the same altitude and area as the triangle.

36. The sum of the distances of any point within a triangle from the corners is less than the perimeter, but greater than half the perimeter of the triangle.

37. Any median of a triangle is less than half the sum of the two sides conterminous with it.

38. The sum of the three medians of a triangle is less than the perimeter, but greater than three-fourths of the perimeter of the triangle.

39. If any point is taken within a triangle and lines drawn from the three corners through it to meet the opposite sides, their sum is greater than half the perimeter of the triangle.

40. If three lines are drawn bisecting the interior angles of a triangle, and terminated by the opposite sides, their sum is less than the perimeter of the triangle.

NOTE—*Deduce from Addenda* (26) (2) *and Ex.* 38.

41. Any point being taken within a quadrilateral, *not* the intersection of its diagonals, the sum of its joins to the corners is greater than the sum of the diagonals.

42. The sum of the four sides of a quadrilateral is greater than the sum, but less than twice the sum of its diagonals.

43. It is possible to draw, to meet within a triangle (not equilateral) two straight lines, terminated in one side, which are together greater than the other two sides, if one of the lines drawn is *not* terminated in a corner of the triangle.

44. If ABCD is a quadrilateral ; M the mid point of BD ; XMY the parallel to AC, meeting the sides in X and Y ; then AY bisects the quadrilateral.

45. If ABCD is a quadrilateral, in which AB, AD are equal ; and CB, CD are equal ; and if M is the mid point of AC ; then MA, MB, MC, MD quadrisect the quadrilateral.

46. Unless one diagonal of a quadrilateral is bisected by the other it is impossible to quadrisect it by joining its four corners to any one point within it.

NOTE—*Use Addenda* (19).

47. From a point within or without a parallelogram, lines are drawn to the extremities of two adjacent sides, and of the diagonal conterminous with them; of the triangles thus formed that whose base is the diagonal is equal to the sum or difference of the other two. Distinguish the cases.

48. If a corner of a square-edged sheet of paper is folded over and creased, and then again folded over and creased, so that the creases are parallel and equidistant, the triangle cut off by the first crease is one-third of the area between the creases.

49. If ABCD is a quadrilateral such that BD, AC make equal angles with DC, and that AC makes the same angle with AD that BD does with BC, then AB is parallel to CD.

50. If a point is taken within a square, the sum of the squares on its joins with the corners is equal to twice the sum of the squares on the perpendiculars dropped from it on the sides.

51. If a point P is joined to the corners A, B, C, D of a rectangle, then

$$PA^2 + PC^2 = PB^2 + PD^2.$$

52. The sum of the perpendiculars from any point within an equilateral triangle on its sides is equal to its altitude.

NOTE—*Use Addenda* (29).

53. The sum of the perpendiculars from any point within an equilateral convex figure is the same wherever the point is taken.

54. If a quadrilateral has a pair of opposite sides equal, and the other pair unequal, then the equal sides are equally inclined to the bisector of the unequal sides.

55. If a rectilineal figure has 100 equal angles, each of them is $176^\circ\ 24'$.

56. If each angle of a rectilineal figure is 162°, the figure has 20 sides.

57. If from any point P within a triangle ABC perpendiculars PX, PY, PZ are dropped on the sides respectively opposite A, B, C; then

$$BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2.$$

58. If a quadrilateral has a pair of parallel sides, the join of their mid points goes through the intersection of the diagonals.

59. The equiangular polygons which will completely fill up the space round a point, only the same kind of polygon being used at once, are those of 3, 4, or 6 sides; but no others will do so.

NOTE—*Use the note to Addenda* (3).

60. ABC is any angle, and AC is perpendicular to BC, *if* P can be found in AC so that, BP being produced to meet the parallel to BC through A in Q, PQ is twice AB, then angle PBC is one-third of angle ABC.

NOTE—*The point* P *cannot be found by the use of an ungraduated ruler and compasses.*

61. The preceding construction for the trisector of any angle is equivalent to the following—From a given point P on the arc of a semi-circle, let a line be drawn meeting the arc again in A, and its chord in B, so that AB is equal to the radius of the semi-circle.

62. If four rods PA, PB, QAC, QBD are pivoted at P, Q, A, B, so as to be capable of angular motion in one plane; and so that PAQB is a parallelogram, and A, B the mid points of QC, QD; then will C, P, D be in a straight line, however the rods are moved about.

63. ABC is any triangle; any parallelograms BADE, BCFG are placed on BA, BC; DE, FG produced meet in H: then the sum of the areas AE, CG is equal to the area of the parallelogram on AC, having sides parallel and equal to BH. (*Pappus' extension of* i. 47.)

64. The area of the equilateral triangle on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the equilateral triangles on its sides.

NOTE—*Let* APB, BQC, CRA *be the* $\triangle^s$, $B\hat{A}C$ *being right. Join* PC, QA; *and drop* $\perp$ PM *on* AB. *Then, easily,* $\triangle$ ABQ = $\triangle$ PBC = $\frac{1}{2}$ $\triangle$ ABC + $\triangle$ APB; *and, similarly,* $\triangle$ ACQ = $\frac{1}{2}$ $\triangle$ ABC + $\triangle$ CRA; *whence, by addition, result follows.*

65. If A and B are the centres of two circles; AP, BQ parallel radii; and R, S the points in which PQ meets the circumferences again; then AR is parallel to BS.

66. If a point is taken *within* a parallelogram, the *sum* of the triangles formed by joining the point with the extremities of a pair of opposite sides is equal to half the parallelogram.

67. If a point is taken *without* a parallelogram, the *difference* of the triangles formed by joining the point with the extremities of a pair of opposite sides is equal to half the parallelogram.

68. If through any point P within a parallelogram ABCD parallels are drawn to the sides, the difference of the parallelograms of which PA, PC are diagonals is equal to twice the triangle PBD.

69. If the mid points of the conterminous sides of two triangles on the same base (either on the same or opposite sides of it) are joined, the joins form a parallelogram - excepting in one case.

70. If from the ends of the base of a triangle perpendiculars are drawn on the external bisector of the vertical angle, and their feet joined transversely to the ends of the base, the joins intersect on the internal bisector of the vertical angle.

71. If in a triangle ABC, BC is greater than BA; D is a point in BC such that BD is half the sum of AB and BC; and E a point in BA produced such that BE is equal to BD; then ED bisects AC.

72. ABC is any triangle; BP, CQ are perpendicular and equal respectively to BA, CA; PM, QN are perpendiculars on BC produced: then—

$$BC = PM + QN, \text{ when angles B and C are each acute;}$$

$$\text{but } BC = PM \sim QN, \text{ when one of them is obtuse.}$$

73. In any quadrilateral the joins of the mid points of opposite sides, and the joins of the mid points of the diagonals, cut in one point.

NOTE—*See Addenda* (18).

74. The bisectors of the *internal* angles of any quadrilateral, form a quadrilateral whose opposite angles are supplementary.

75. The bisectors of the *external* angles of any quadrilateral, form a quadrilateral whose opposite angles are supplementary.

76. If the diagonals of a parallelogram cut in O, and P is any point within the triangle AOB; then

$$\triangle CPD = \triangle APB + \triangle APC + \triangle BPD.$$

77. A pair of the diagonals of parallelograms, which are about the diagonal of a parallelogram, are parallel.

78. If a parallel is drawn to the hypotenuse of a right-angled triangle, and terminated by the sides; and if its ends are joined to the vertices of the acute angles; then the sum of the squares on the joins is equal to the square on the hypotenuse together with the square on the parallel to it.

79. If X and Y are the respective mid points of the sides AB, AC of any triangle; and if BY, CX cut in O; then triangle AXY is equal to three times triangle XOY.

80. If a line PMQ is drawn through the mid point M of the side BC of a triangle ABC, so as to cut off equal parts AP, AQ from AB, AC (produced if necessary) respectively; then BP is equal to CQ.

81. If on the sides AB, BC, CD of a parallelogram ABCD, equilateral triangles are described—that on BC towards the same parts as the parallelogram, and those on AB, CD towards opposite parts—then the distances of the vertices of the triangles on AB, CD, from that on BC, are respectively equal to the diagonals of the parallelogram.

82. If two triangles ABC, abc, have their sides, distinguished by corresponding letters, parallel (ABC being the smaller) and if BP, CX are perpendicular to bc ; CQ, AY to ca ; and AR, BZ to ab ; then—

$$AP^2 + BQ^2 + CR^2 = AX^2 + BY^2 + CZ^2.$$

NOTE—*Drop* $\perp$s *from* A, B, C, *on* bc, ca, ab ; *and use Ex.* 57.

83. If a quadrilateral has a pair of opposite sides equal, and a pair of opposite angles equal, *and* obtuse, it is a parallelogram ; but if the pair of opposite equal angles are acute, not necessarily so.

NOTE—*See Addenda* (9).

84. If through the corners A, B, C of a triangle, parallels are drawn meeting the opposite sides (produced if necessary) respectively in X, Y, Z ; then

$$\triangle AYZ = \triangle BZX = \triangle CXY = \triangle ABC.$$

85. In the figure of i. 47—

(1) AX, QC are $\perp$;

(2) $\triangle QBX = \triangle NCY = \triangle PAM = \triangle ABC$;

(3) AD, QC, BN are concurrent ;

(4) the medians of $\triangle ABC$ are $\perp$ to PM, QX, YN respectively ; and each median is half the line to which it is $\perp$;

(5) if QH, AL, NK are $\perp$s dropped on direction BC,

$$\triangle BHQ \equiv \triangle ALB, \text{ and } \triangle CKN \equiv \triangle ALC ;$$

(6) Q, A, N are collinear ;

(7) if XE, YF are $\perp$s dropped on directions QB, NC,

$$\triangle EBX \equiv \triangle ABC \equiv \triangle FYC ;$$

(8) $QX^2 + NY^2 = 5\,BC^2$;

(9) sum of squares on sides of rectilineal figure XYNMPQ $= 8\,BC^2$;

(10) if QC cut AB in U, and BN cut AC in V, AU = AV ;

(11) if AX, QC cut in W, BW bisects $Q\hat{W}X$.

86. (1) (2) (3) (4) (5) (11) of 85, are also true when $B\hat{A}C$ is *not* right.

87. If ABCD is a quadrilateral ; X, Y the respective mid points of AC, BD ; and O their intersection ; then—

$$4\,\triangle XOY = (\triangle AOB + \triangle COD) \sim (\triangle AOD + \triangle COB).$$

88. If a pair of opposite sides of a quadrilateral meet in O ; and X, Y are the mid points of its diagonals ; then the quadrilateral is equal to four times the triangle XOY.

89. The following construction will quadrisect any quadrilateral ABCD—From X, Y, the mid points of AC, BD, draw XO, YO parallel to BD, AC; then the joins of O to the mid points of AB, BC, CD, DA, will quadrisect ABCD.

NOTE—*Join* X *to the mid points of two adjacent sides, and use Addenda* (17) *and* i. 37.

90. The following dissection will cut a square into two others, and incidentally prove i. 47—On each side of the square, as hypotenuse, describe equal right-angled triangles, so as to lie on the square, and have the shorter and longer sides of each similarly situated with regard to the hypotenuse; then cutting along the sides of the triangles will give pieces that will form the two squares on the sides of one triangle.

NOTE—*This (the Hindoo) is said to be the oldest of all the dissecting proofs.*

91. The following construction will bisect a triangle ABC by a line through a given point P in BC—Take the median AM, and draw MQ parallel to AP to meet a side of the triangle in Q: PQ is the bisector.

92. The points P, Q, in which a line AB is trisected, may be found by any one of the following constructions—

(α) Draw any line from A, and in it take X, Y, Z, so that AX, XY, YZ are equal: join BZ, and draw XP, YQ parallel to BZ, to meet AB in P, Q.

(β) Describe an equilateral triangle ABC on AB: let the bisectors of angles A, B, meet in O; and draw OP, OQ parallel to AC, BC to meet AB in P, Q.

(γ) Describe any triangle ABC on AB: draw the median AM: join C to the mid point of AM, and produce this join to meet AB in P; and draw MQ parallel to CP.

93. If a point O, and its distances a, b, c from the corners of an equilateral triangle, within which it lies, are given, the triangle can be constructed thus—Make triangle OXC so that OX, XC, OC are respectively equal to a, b, c; and construct an equilateral triangle OAX on OX; and join AC: then the equilateral triangle on AC is the one wanted.

94. If two opposite angles of a quadrilateral are equal, the bisectors of the remaining angles are either parallel, or coincident.

95. If ABC is an isosceles triangle, in which the angle at C is right, and V is any point in AB; then $AV^2 + BV^2 = 2\,CV^2$.

96. If ABC, A′B′C′ are triangles whose correspondingly lettered sides are equal and parallel; then of the parallelograms formed by joining correspondingly lettered corners, one is equal to the sum of the other two: hence may be deduced a proof of i. 47.

BOOK ii.

Def. A right-angled parallelogram is called a **rectangle.**

Note (1)—From the properties of a parallelogram, we see that if—

1°, one angle of a parallelogram is right it is a rectangle;

2°, the two lines which contain an angle of one rectangle are equal to the two which contain an angle of another rectangle, each to each, the rectangles are identically equal.

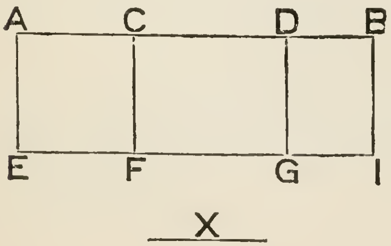
Since therefore a rectangle depends only on the lengths of a pair of adjacent sides, it is said to be '*under*,' or '*contained by*,' two of the sides which form any one of its angles. The words '*rectangle under* AB *and* CD' will be abbreviated into '*rect. under* AB, CD,' in Euclid's Props.; and still further abbreviated into 'AB . CD' in the *Addenda* and *Exercises*.

Note (2)—If to each of two lines at right angles are drawn any number of parallels, *all* the quadrilaterals then formed are rectangles.

Note (3)—An equilateral rectangle is a square.

Proposition 1.

THEOREM—*If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.*



Let AB be the one line, divided in C and D; and X the other.

Draw AE so that it is ⊥, and equal to AB.

Draw $\|^s$ to AE, through C, D, B, meeting the $\|$ to AB, through E, in pts. F, G, I respectively. Then all the quads. in the fig. are formed by $\|^s$ to two $\perp$ lines, and $\therefore$ are rects.

And since X = AE = CF = DG = BI,

$\therefore$ rects. AI, AF, CG, DI are respectively contained by

AB and X, AC and X, CD and X, DB and X.

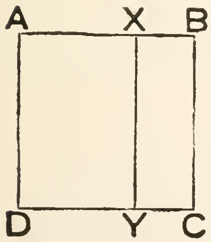
But fig. AI = sum of figs. AF, CG, DI:

i.e. rect. under AB, X

= rect. under AC, X + rect. under CD, X + rect. under DB, X.

Proposition 2.

THEOREM—*If a straight line is divided into two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.*



Let AB be a st. line divided in X.

On AB describe sq. ABCD.

Draw XY $\|$ to AD, and meeting DC in Y. Then the figs. AY, BY have their sides $\|$ to the sides of a sq., and $\therefore$ are rects.

And since AD = AB,

$\therefore$ fig. AY = rect. under AB, AX.

Also since BC = BA,

$\therefore$ fig. BY = rect. under AB, BX.

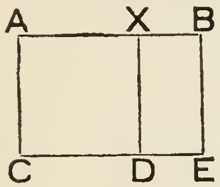
But figs. AY and BY make up fig. AC :

i. e. rect. under AB, AX + rect. under AB, BX = sq. on AB.

Note—Prop. 2 is that case of Prop. 1 when the divided and undivided lines are equal.

Proposition 3.

THEOREM—*If a straight line is divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.*



Let AB be a st. line divided in X.

On AX describe sq. ACDX.

Draw BE ∥ to XD; and meeting CD produced in E.

Then the figs. AE, XE are formed by ∥s to the sides of a sq., and ∴ are rects.

And since AC = AX,

∴ fig. AE = rect. under AB, AX.

Also since XD = AX,

∴ fig. XE = rect. under AX, XB.

But fig. AE = sum of figs. AD, XE:

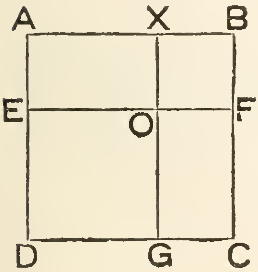
i. e. rect. under AB, AX = sq. on AX + rect. under AX, XB.

Note (1)—Prop. 3 is that case of Prop. 1 when the undivided line is equal to one of the parts of the divided line.

Note (2)—If in Props. 2 and 3 we consider AB, AX as two lines, so that BX is their difference, then the two Props. may be included in this single enunciation—*The difference between the rectangle under two lines and the square on one of them is equal to the rectangle under that one and their difference.*

Proposition 4.

THEOREM—*If a straight line is divided into two parts, the square on the whole line is equal to the sum of the squares on the parts, together with twice the rectangle contained by the parts.*



Let st. line AB be divided in X.
On AB describe sq. ABCD.
Take E in AD so that AE = BX,
∴ ED = AX.

Draw XOG, EOF ∥ to sides of sq., and meeting DC, BC respectively in G, F,

Then all the quads. in the fig. are formed by ∥s to the sides of a sq., and ∴ are rects.

And BF = AE = BX.

∴ fig. BO, being equilat., is a sq.; and it is on BX.

Similarly fig. DO is a sq.; and is on EO, which = AX.

Again since OF = OX, and OG = OE,

∴ fig. CO = fig. AO.

Also, since XO = BX,

∴ fig. AO = rect. under AX, BX.

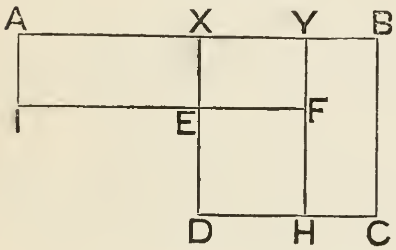
But fig. AC = sum of figs. BO, DO, CO, AO:

i. e.

sq. on AB = sq. on AX + sq. on BX + 2 rect. under AX, BX.

Proposition 5.

THEOREM—*If a straight line is divided equally and also unequally, the rectangle under the unequal parts, and the square on the part between the points of section are together equal to the square on half the line.*



Let st. line AB be divided equally in X, unequally in Y.

On BX describe sq. BCDX.

In XD take E so that XE = YB.

∴ ED = XY.

Draw a || to AB through E, meeting ||s to BC through Y and A in pts. F and I respectively; and let YF meet DC in H.

Then all the quads. in the fig. are formed by ||s to the sides of a sq., and ∴ are rects.

Now since AX = BC, and AI = XE = YB,

∴ AE = YC.

∴ YC + XF = AF,

= rect. under AY, YB, ∵ AI = YB.

Also since ED = XY = EF,

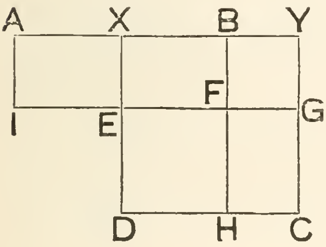
∴ EH is a sq.; and it is on EF, which = XY.

But figs. YC, XF, EH make up fig. XC:

i.e. rect. under AY, YB + sq. on XY = sq. on BX.

Proposition 6.

THEOREM—*If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on the half of the bisected line, is equal to the square on the straight line made up of the half and the part produced.*



Let st. line AB be bisected in X, and produced to Y.

On XY describe sq. XYCD.

In XD take E so that

$$XE = BY.$$

$$\therefore \quad ED = XB.$$

Through E draw a || to AY, meeting YC in G, and ||s to YC, through B and A in F, I respectively; and let BF meet DC in H.

Then all the quads. in the fig. are formed by ||s to the sides of a sq., and ∴ are rects.

Now since $FG = BY = XE$,

and $FH = ED = BX = AX$,

$\therefore \quad FC = AE.$

$\therefore \quad FC + XG = AG,$

= rect. under AY, YB, $\because YG = XE = BY.$

Also $ED = XB = EF.$

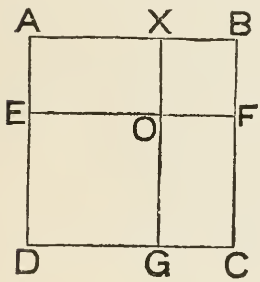
∴ EH is a sq.; and it is on EF, which = XB.

But figs. FC, XG, EH make up fig. XC:

i.e. rect. under AY, YB + sq. on XB = sq. on XY.

Proposition 7.

THEOREM—*If a straight line is divided into any two parts, the square on the whole line together with the square on one of the parts is equal to twice the rectangle contained by the whole line and that part, together with the square on the remaining part.*



Let st. line AB be divided in X.
On AB describe sq. ABCD.
In AD take E so that AE = BX.
$\therefore$ ED = AX.

Draw XOG, EOF $\parallel$ to sides of sq., and meeting CD, CB respectively in G, F.

Then all the quads. in the fig. are formed by $\parallel^s$ to the sides of a sq., and $\therefore$ are rects.

And BF = AE = BX.

$\therefore$ fig. BO, being equilat., is a sq.; and it is on BX.

Similarly fig. DO is a sq.; and is on EO, which = AX.

Also since BF = BX,

$\therefore$ fig. AF = rect. under AB, BX.

And since BC = AB,

$\therefore$ fig. XC = rect. under AB, BX.

But figs. BD and BO make up figs. AF, XC, DO:

i. e.

sq. on AB + sq. on BX = 2 rect. under AB, BX + sq. on AX.

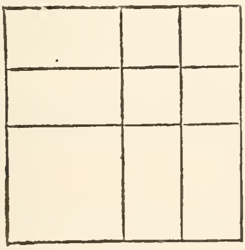
Proposition 8.

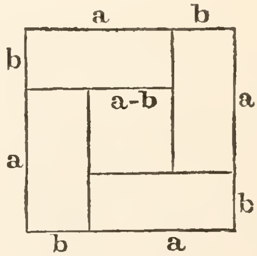
THEOREM—*If a straight line is divided into any two parts, four times the rectangle under the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first part.*

This proposition is omitted because—

1°, never made use of by Euclid;
2°, of very little use;
3°, very lengthy;
4°, really contained in ii. 5 and 6; see ii. *Addenda* (3).

The student may work it as an exercise, in the same way as Props. 9 and 10 are done here; or graphically from either of these figs.





Note—The manner in which the right-hand fig. applies to the Prop. is indicated by this algebraic identity, which is its analogue—

$$4\,ab + (a-b)^2 \equiv (a+b)^2.$$

On p. 118 will be found a proof in which the Prop. is shown to be an easy deduction from Props. 3 and 4.

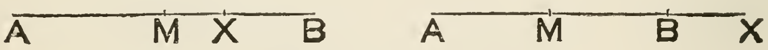
Def. A point taken in a finite straight line is said to divide the line **internally,** or simply, to divide it; and, if the line is produced, a point taken on the produced part, is said (by analogy) to divide the line **externally**; in either case the distances of the point from the extremities of the finite line are called the **segments** of the line.

Note—It follows, from this definition, that a straight line is equal to the sum or difference of its segments, according as it is divided internally or externally.

Propositions 9 and 10.

THEOREM—*If a straight line is divided internally or externally at any point, the sum of the squares on the segments is double the sum of the squares on half the line and on the line between the point of division and the middle point of the line.*

(1) (2)



Let AB be a st. line whose mid pt. is M.

Let it be divided in X, internally fig. (1), or externally fig. (2).

Then, by ii. 4, we have

sq. on AX = sq. on AM + sq. on MX + 2 rect. under AM, MX.

And, by ii. 7, we have

sq. on BX + 2 rect. under BM, MX = sq. on BM + sq. on MX.

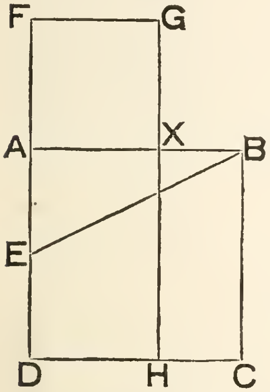
∴, putting AM for BM, adding corresponding sides, and omitting 2 rect. AM, MX from each side, we get

sq. on AX + sq. on BX = 2 (sq. on AM + sq. on MX).

Note—It is clear that Props. 2 to 10 of Book ii are merely amplifications of Prop. 1, and that they can be directly deduced from it. As a good exercise, the learner should make the deduction in each case.

Proposition 11.

PROBLEM—*To divide a given straight line into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.*



Let AB be the given st. line.
On AB describe the sq. ABCD.
Bisect AD in E; and join EB.
Produce EA to F so that

EF = EB.

On the same side of AF as AB complete the sq. AFGX, of which AF is a side.

Then, the $\angle$s FAX, DAB being rt., AX will lie along AB; and since EA + AB > EB, i.e. > EA + AF

$\therefore$ AB > AX

so that X will lie in AB.

Produce GX to meet DC in H.

Since AD is bisected in E and produced to F,

$\therefore$ rect. under DF, FA + sq. on AE = sq. on EF,
= sq. on EB,
= sq. on AB + sq. on AE.

Take sq. on AE from each side, and
rect. under DF, FA = sq. on AB.

But fig. FH = rect. under DF, FA, $\because$ FG = FA.

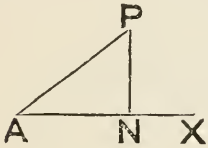
$\therefore$ fig. FH = fig. AC.

Take fig. AH from each.

Then fig. AG = fig. XC,
= rect. under AB, BX, $\because$ BC = AB:

i.e. sq. on AX = rect. under AB, BX.

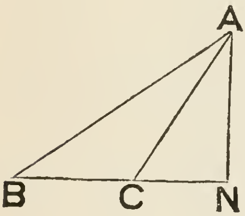
Def. When we speak of the **projection** of a terminated straight line on another straight line, we mean the distance between the feet of perpendiculars drawn from the extremities of the terminated line on the other. Also, by the **projection** of a point on a line is meant the foot of the perpendicular from the point on the line.



Note—If AP is a terminated line, and AX any other line through A; then, if PN is perpendicular to AX, AN is the *projection* of AP on AX; and N is the projection of P on AX.

Proposition 12.

THEOREM—*In an obtuse-angled triangle the square on the side opposite the obtuse-angle is greater than the sum of the squares on the sides forming the obtuse angle, by twice the rectangle contained by either of these latter sides, and the projection of the other upon it.*



Let ABC be a $\triangle$ in which $A\hat{C}B$ is obtuse.

Draw AN $\perp$ to BC produced; so that CN is the projection of AC on BC.

Then, by ii. 4, we have

sq. on BN = sq. on BC + sq. on CN + 2 rect. under BC, CN.

$\therefore$, adding sq. on AN to each side; and recollecting that

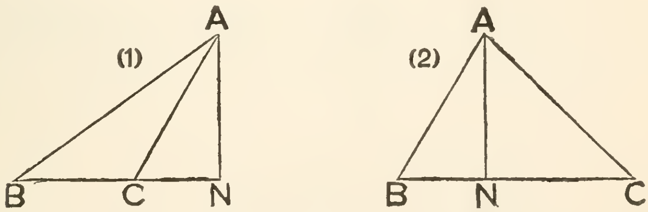
$$\left.\begin{array}{r}\text{sq. on BN} + \text{sq. on AN} = \text{sq. on AB,}\\ \text{and sq. on CN} + \text{sq. on AN} = \text{sq. on AC,}\end{array}\right\} \because \hat{N} \text{ is right,}$$

we get

sq. on AB = sq. on BC + sq. on AC + 2 rect. under BC, CN.

Proposition 13.

THEOREM—*In any triangle the square on a side opposite an acute angle is less than the sum of the squares on the sides forming the acute angle, by twice the rectangle contained by one of the latter lines and the projection of the other upon it.*



Let ABC be a $\triangle$ in which $\hat{B}$ is acute.

Draw AN $\perp$ to BC, which may need to be produced, as in fig. (1), or not, as in fig. (2): then BN is the projection of AB on BC.

Now $\because$ BN is divided in C in fig. (1),
or BC „ N in fig. (2);
$\therefore$, in both cases, by ii. 7, we have

sq. on BC + sq. on BN = sq. on CN + 2 rect. under BC, BN.

$\therefore$, adding sq. on AN to each side; and recollecting that

$$\left.\begin{aligned} &\text{sq. on BN + sq. on AN = sq. on AB,} \\ \text{and } &\text{sq. on CN + sq. on AN = sq. on AC,} \end{aligned}\right\} \because \hat{N} \text{ is right,}$$

we get

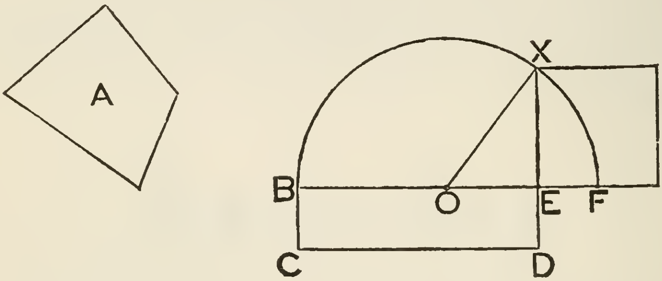
sq. on BC + sq. on AB = sq. on AC + 2 rect. under BC, BN:

i. e.

sq. on AC < sq. on BC + sq. on AB by 2 rect. under BC, BN.

Proposition 14.

PROBLEM—*To describe a square whose area shall be equal to the area of a given rectilineal figure.*



Let A be the given rectil. fig.

Make a rect. BCDE, equal to A.

Then, if BE = BC, BD is a sq., and problem is solved.

Assume BE and BC unequal.

Produce BE to F, so that EF = ED.

Bisect BF in O; and with O as centre, and BO as radius, describe a ⊙.

Produce DE to meet this ⊙ in X; and join OX.

Since BF is divided equally in O, and unequally in E,

∴ rect. under BE, EF + sq. on OE = sq. on BO,

= sq. on OX,

= sq. on EX + sq. on OE.

Take sq. on OE from each side.

Then rect. under BE, EF = sq. on EX:

i.e. the sq. described on EX = fig. A.

ADDENDA TO BOOK ii.

Corollaries to the Props. in Book ii.

ii. 1. (α) If two lines are each divided into any number of parts, the rectangle under the lines is equal to the sum of the rectangles formed by each part of one line with each part of the other.

(β) The rectangle under a line and the difference of two other lines, is equal to the difference of the rectangles under the first line and each of the others.

ii. 4. (α) The square on a line is equal to four times the square on its half.

(β) If a line is divided into any number of parts the square on the whole line is equal to the sum of the squares on its parts, together with twice the sum of the rectangles formed by each part with each other part.

Def. A line divided as in ii. 11, is said to be divided in *medial section.*

ii. 11. (α) In the given construction DA is produced to F so that the rectangle under the whole line produced (DF) and the part produced (AF) is equal to the square on the original line (AD): that is DA is produced to F so that FD is divided in A in *medial section.*

(β) Also since FD is divided in A in the same way as AB in X; and, if Y is taken in AD so that AY is equal to AF, then AD, AY being respectively equal to AB, AX, it follows that AD is divided in Y in the same way as FD in A—from all this we see that—If a line is divided in *medial section,* and a part is taken in the greater segment so as to be equal to the lesser segment, the greater segment will be divided in *medial section.*

Note—The term *medial section* may be extended so as to include the *external* division of a line (see *def.* on p. 94) thus.

Def. A line is said to be divided in *medial section*—

1°, *internally* (constructed on p. 95) when
rect. under original line and *lesser* segt. = sq. on *greater* segt.;

2°, *externally* (constructed on p. 103) when
rect. under original line and *greater* segt. = sq. on *lesser* segt.

ii. 14. The process of finding a square which is equal to the area of a given figure is called the *quadrature* of the figure. This Proposition is the fourth step in the quadrature of a given *rectilineal* figure—the preceding steps being i. 42, 44, 45.

SOME IMMEDIATE DEVELOPMENTS OF THE PROPS. IN BOOK ii.—NOT SO OBVIOUS AS TO BE PROPERLY CALLED COROLLARIES.

THEOREM (1)—*The difference of the squares on two lines is equal to the rectangle under their sum and difference.*

Let AB, AC be the lines, placed so that AC the lesser is conterminous, and in the same direction, with AB.

$$\text{Then} \quad AB^2 = AB \, . \, AC + AB \, . \, CB.$$

$$\text{And } AB \, . \, AC = AC^2 + AC \, . \, CB.$$

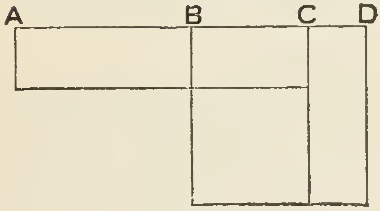
$\therefore$ adding, and omitting $AB \, . \, AC$ from each side, we get

$$AB^2 = AC^2 + AB \, . \, CB + AC \cdot CB,$$

$$\text{or } AB^2 - AC^2 = (AB + AC)\, CB,$$

$$= (AB + AC)(AB - AC).$$

Note—The last Theorem may be deduced at once from ii. 5, or from ii. 6; or it may be done graphically, from the annexed figure, in which AB, BC are the given lines; BC the lesser, is produced to D so that BD = BA; and fig. is completed as in ii. 5.



The learner should write out the complete construction and proof.

THEOREM (2)—*The distance of the mid point of a finite straight line from a point of internal division is half the difference of the segments; and its distance from a point of external division is half the sum of the segments.*

Let AB be a finite straight line, divided internally in P, and externally in Q: take M the mid pt. of AB.

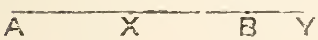
$$\begin{aligned}
\text{Then } PM &= PA - AM, \quad \text{if } P \text{ lie in } BM. \\
&= PA - BM, \\
&= PA - PB - PM, \\
\therefore \quad 2\,PM &= PA - PB. \\
\text{Similarly} \quad 2\,PM &= PB - PA, \quad \text{if } P \text{ lie in } AM. \\
\therefore \quad \text{always } PM &= \tfrac{1}{2}\,(PA \sim PB). \\
\text{Again } QM &= QB + BM, \\
&= QB + AM, \\
&= QB + QA - QM, \\
\therefore \quad 2\,QM &= QB + QA, \\
\text{or } QM &= \tfrac{1}{2}\,(QA + QB).
\end{aligned}$$

THEOREM (3)—*The rectangle under two lines, together with the square on half their difference, is equal to the square on half their sum.*

This is merely ii. 5 and ii. 6 included in one enunciation.

For if AB is a line divided equally in X, and unequally in Y.

$$\begin{aligned}
\text{Then, by ii. 5,} \quad AY \,.\, YB + XY^2 &= BX^2. \\
\text{But } XY &= \tfrac{1}{2}\,(AY \backsim BY). \\
\text{And } BX &= \tfrac{1}{2}\,(AY + BY).
\end{aligned}$$



Again, if AB is divided equally in X and produced to Y.

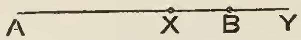
$$\begin{aligned}
\text{Then, by ii. 6,} \quad AY \,.\, YB + BX^2 &= XY^2. \\
\text{But } BX &= \tfrac{1}{2}\,(AY - BY). \\
\text{And } XY &= \tfrac{1}{2}\,(AY + BY).
\end{aligned}$$

$\therefore$ for any two lines AY, BY we have

$$AY \,.\, YB + \left(\frac{AY \sim BY}{2}\right)^2 = \left(\frac{AY + BY}{2}\right)^2.$$

This Theorem is also a modified way of enunciating ii. 8.

For let AB be divided into any two parts in X.



Produce AB to Y, so that $BY = BX$.

Then, by ii. 8, $4 AB \cdot BX + AX^2 = AY^2$.

But $AX = AB - BX$.

And $AY = AB + BX$.

$\therefore$ again, for any two lines AB, BX, we have

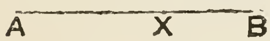
$$AB \,.\, BX + \left\{\frac{AB - BX}{2}\right\}^2 = \left\{\frac{AB + BX}{2}\right\}^2.$$

Note—We thus see that ii. 8 is really included in ii. 5 and ii. 6.

THEOREM (4)—*The square on the sum of two lines is greater than the sum of the squares on those lines by twice the rectangle under them.*

This is only a modified enunciation of ii. 4.

THEOREM (5)—*The square on the difference of two lines is less than the sum of the squares on them by twice the rectangle under them.*



This is another way of enunciating ii. 7. For if AB is a line divided into any two parts in X.

Then, by ii. 7, $AB^2 + BX^2 = 2 AB \,.\, BX + AX^2$.

or $AX^2 = AB^2 + BX^2 - 2 AB \,.\, BX$,

or $(AB - BX)^2 = AB^2 + BX^2 - 2 AB \,.\, BX$,

which is the Theorem.

Theorem (6)—*The square on the sum of two lines, together with the square on their difference, is equal to twice the sum of the squares on the lines.*

This is only another way of including ii. 9 and 10 in one enunciation.

(1)

(2)

For if AB be a line, whose mid pt. is M; and X a pt. either of internal division, fig. (1); or external division, fig. (2), then, by ii. 9 and 10, we have, in both cases,

$$AX^2 + BX^2 = 2\,AM^2 + 2\,MX^2,$$

$$\text{or} \quad 4\,AM^2 + 4\,MX^2 = 2\,(AX^2 + BX^2),$$

$$\text{or} \quad (AX + BX)^2 + (AX \backsim BX)^2 = 2\,(AX^2 + BX^2).$$

Note—This Theorem is also an immediate deduction from ii. 4 and ii. 7 taken together.

THEOREM (7)—*The sum of the squares on any two lines is equal to twice the square on half their sum together with twice the square on half their difference.*

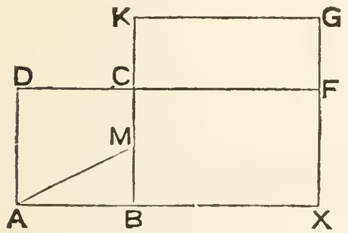
This again is only another way of enunciating ii. 9 and 10 together.

For, as in preceding Theorem,

$$AX^2 + BX^2 = 2\,AM^2 + 2\,MX^2.$$

$$\therefore \quad AX^2 + BX^2 = 2\left\{\frac{AX + BX}{2}\right\}^2 + 2\left\{\frac{AX \sim BX}{2}\right\}^2.$$

Extension of ii. 11—*the external division of a line in medial section.*



On given line AB describe sq. ABCD.

Join M (the mid pt. of BC) to A.

Produce BC to K, so that MK = MA.

On BK describe sq. BKGX, which will have a side BX in same direction as BA.

Produce DC to meet XG in F.

Then, since BC is bisected in M, and produced to K,

$$BK \,.\, KC + BM^2 = MK^2 = MA^2 = AB^2 + BM^2.$$

$$\therefore \quad BK \,.\, KC = AB^2:$$

i.e. fig. CG = fig. AC, since KG = BK.

Add fig. BF to each.

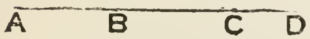
Then fig. BG = fig. AF:

i.e. $BX^2 = AB \,.\, AX$.

Some useful Theorems depending on Book ii.

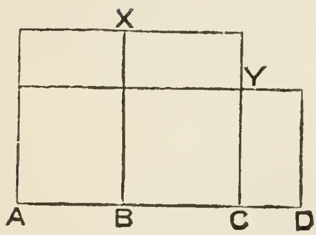
Theorem (8) *(Euler's)—If four points* A, B, C, D, *are in order on the same line, then*

$$AC \,.\, BD = AB \,.\, CD + BC \,.\, AD.$$



By repeated applications of ii. 1, we get

$$\begin{aligned} AC \,.\, BD &= (AB + BC)\, BD, \\ &= AB \,.\, BD + BC \,.\, BD, \\ &= AB\,(CD + BC) + BC \,.\, BD, \\ &= AB \,.\, CD + BC \,.\, AB + BC \,.\, BD, \\ &= AB \,.\, CD + BC\,(AB + BD), \\ &= AB \,.\, CD + BC \,.\, AD. \end{aligned}$$



Or Theorem may be proved graphically, from annexed fig. in which

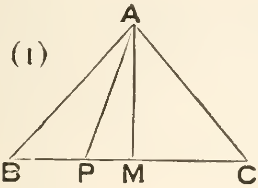
BX is $\perp$ to AD, and = BD,

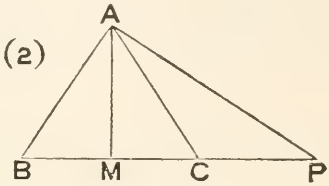
CY is $\perp$ to AD, and = CB.

Then rects. are completed.

The learner should write out the proof.

THEOREM (9)—*If from the vertex of an isosceles triangle any straight line is drawn to meet the base internally, or externally, the difference of the squares on one of the sides and on the line so drawn is equal to the rectangle under the segments of the base.*





Let ABC be a $\triangle$; in which $AB = AC$.

And let BC be divided in P, internally fig. (1), externally fig. (2).

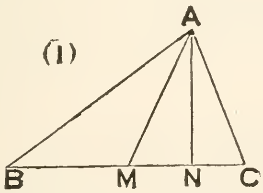
Draw $AM \perp$ to BC.

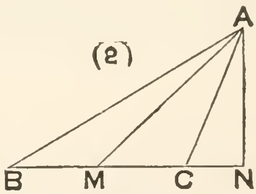
Then $AB^2 \sim AP^2 = BM^2 \sim PM^2$, *Cor.* to i. 47

$= (BM + PM)(BM \sim PM)$,

$= BP \cdot CP$ in both figs.

THEOREM (10)—*The difference of the squares on two sides of a triangle is equal to twice the rectangle under the base and the projection of the median, bisecting the base, on the base.*





Let ABC be a $\triangle$; M the mid pt. of its base BC.

$AN \perp$ to BC, fig. (1), or BC produced, fig. (2).

In fig. (1) $2\,MN = BN \sim CN$.

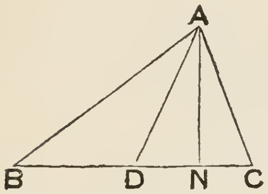
In fig. (2) $2\,MN = BN + CN$.

Also $AB^2 \sim AC^2 = BN^2 \sim CN^2$, *Cor.* to i. 47

$= (BN + CN)(BN \sim CN)$,

$= 2\,BC \cdot MN$, in both figs.

THEOREM (11)—*In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects that side.*



In $\triangle$ ABC let AD be the median from A.

Draw AN $\perp$ to BC.

Then, unless AN coincides with AD, one of the $\wedge^s$ at D (say ADB) is obtuse, and the other ADC is acute.

$$\therefore \quad AB^2 = AD^2 + BD^2 + 2\,BD\,.\,DN.$$

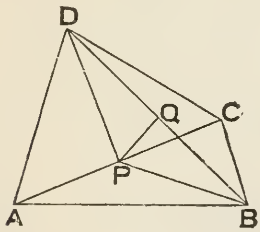
$$\text{And } AC^2 + 2\,CD\,.\,DN = AD^2 + DC^2.$$

$\therefore$, adding corresponding sides, putting BD for DC, and omitting $2\,BD\,.\,DN$ from each side, we get

$$AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2.$$

In the case when AN coincides with AD the Theorem is an immediate deduction from i. 47.

THEOREM (12)—*The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the join of the mid points of its diagonals.*



Let ABCD be a quad.

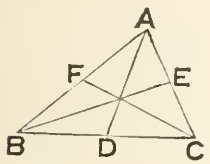
P, Q the respective mid pts. of AC, BD.

Join PB, PD.

$$\begin{aligned} \text{Then } AB + BC^2 &= 2\,BP^2 + 2\,AP^2, \\ AD^2 + CD^2 &= 2\,DP^2 + 2\,AP^2, \\ \therefore \quad AB^2 + BC^2 + CD^2 + DA^2 &= 2\,(BP^2 + DP^2) + 4\,AP^2, \\ &= 4\,PQ^2 + 4\,BQ^2 + 4\,AP^2, \\ &= 4\,PQ^2 + BD^2 + AC^2. \end{aligned}$$

Cor. An important particular case is that in which the quadrilateral is a parallelogram, when the Theorem becomes—*The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.*

THEOREM (13)—*In any triangle three times the sum of the squares on its three sides is equal to four times the sum of the squares on its three medians.*



Let AD, BE, CF be the medians of the $\triangle$ ABC.

$$\text{Then } AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2.$$

$$\therefore\ 2\,AB^2 + 2\,AC^2 = 4\,AD^2 + 4\,BD^2,$$

$$= 4\,AD^2 + BC^2.$$

And two similar results.

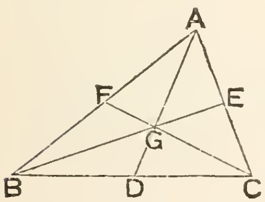
$\therefore$, adding the three, and omitting $AB^2 + BC^2 + CA^2$ from each side, we get

$$3\,(AB^2 + BC^2 + CA^2) = 4\,(AD^2 + BE^2 + CF^2).$$

Cor. In an equilat. $\triangle$,

the sq. on an alt. = three-fourths the sq. on a side.

THEOREM (14)—*In any triangle three times the sum of the squares on the distances of the centroid from the three corners is equal to the sum of the squares on the three sides.*



Let AD, BE, CF be the medians of $\triangle$ ABC; G its centroid.

$$\text{Then } BG^2 + CG^2 = 2\,GD^2 + 2\,BD^2.$$

$$\therefore\ 2\,BG^2 + 2\,CG^2 = 4\,GD^2 + 4\,BD^2.$$

$$= AG^2 + BC^2.$$

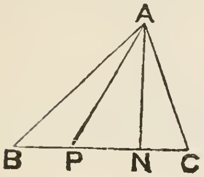
And two similar results.

$\therefore$, adding the three, and omitting $AG^2 + BG^2 + CG^2$ from each side, we get

$$3\,(AG^2 + BG^2 + CG^2) = AB^2 + BC^2 + CA^2.$$

THEOREM (15)—*If* P *is that point in side* BC *of triangle* ABC *for which* BP *is half* CP, *then*

$$2\,AB^2 + AC^2 = 6\,BP^2 + 3\,AP^2.$$



Draw AN $\perp$ to BC.

Then, unless AN coincides with AP, one of the $\wedge^s$ at P (say APB) is obtuse, and the other APC acute.

$$\therefore \quad AB^2 = BP^2 + AP^2 + 2\,BP \,.\, PN,$$

$$\text{or} \quad 2\,AB^2 = 2\,BP^2 + 2\,AP^2 + 2\,CP \,.\, PN.$$

$$\text{And} \quad AC^2 = CP^2 + AP^2 - 2\,CP \,.\, PN,$$

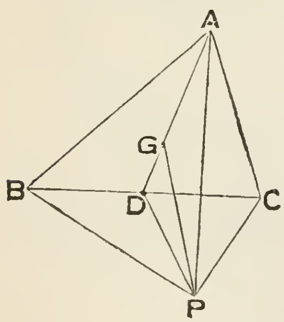
$$= 4\,BP^2 + AP^2 - 2\,CP \,.\, PN.$$

$\therefore$, adding corresponding sides, we get

$$2\,AB^2 + AC^2 = 6\,BP^2 + 3\,AP^2.$$

In the case when AN coincides with AP the Theorem is an immediate deduction from i. 47.

THEOREM (16)—*The sum of the squares on the distances of the three corners of a triangle from any point is equal to the sum of the squares on their distances from the centroid, together with three times the square on the distance between the point and the centroid.*



Let ABC be a $\triangle$; AD a median; and G its centroid.

Let P be any pt. joined to A, B, C, D, G.

Then, since AG = 2 DG,

$$PA^2 + 2\,PD^2 = 6\,GD^2 + 3\,PG^2.$$

$$\text{Also } PB^2 + PC^2 = 2\,PD^2 + 2\,BD^2.$$

$\therefore$, adding and omitting $2\,PD^2$ from each side, we get

$$PA^2 + PB^2 + PC^2 = 4\,GD^2 + 2\,GD^2 + 2\,BD^2 + 3\,PG^2,$$

$$= AG^2 + BG^2 + CG^2 + 3\,PG^2.$$

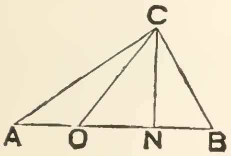
Cor. When ABC is equilat., G is the centre of the $\triangle$ (see p. 74); and then

$$PA^2 + PB^2 + PC^2 = 3(AG^2 + PG^2).$$

THEOREM (17)—*If the side* AB, *of a triangle* ABC, *is divided into* $m + n$ *equal parts; and if* O *is the point in* AB *for which* AO *contains* n *parts and* BO m *parts; then—*

$$m \cdot AC^2 + n \cdot BC^2 = (m + n) \cdot OC^2 + m \cdot AO^2 + n \cdot BO^2,$$

the dots denoting multiplication.



If a line XY = length of each part,

then $AO = n \cdot XY$,

$BO = m \cdot XY$,

$\therefore\ m \cdot AO = mn \cdot XY = n \cdot BO$.

Draw CN $\perp$ to AB, then if N is in BO

$\widehat{COB}$ is acute, and $\widehat{COA}$ obtuse.

$$\therefore\ m \cdot AC^2 = m \cdot (OC^2 + AO^2 + 2\,AO \cdot ON).$$

$$\text{And } n \cdot BC^2 = n \cdot (OC^2 + BO^2 - 2\,BO \cdot ON).$$

$\therefore$, adding, and recollecting that $m \cdot AO - n \cdot BO = 0$, we get

$$m \cdot AC^2 + n \cdot BC^2 = (m + n) \cdot OC^2 + m \cdot AO^2 + n \cdot BO^2.$$

Note (1)—Theorems (11) and (15) are evidently particular cases of this.

Note (2)—Since $AO = \frac{n}{m + n} \cdot AB$, and $BO = \frac{m}{m + n} \cdot AB$, the above result may be written

$$m \cdot AC^2 + n \cdot BC^2 = (m + n) \cdot OC^2 + \frac{mn}{m + n} \cdot AB^2.$$

And, if ACB is a right $\angle$, this becomes (*Euler*)

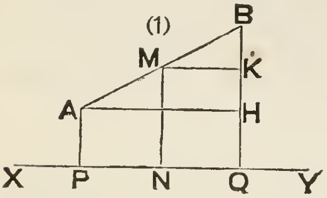
$$m^2 \cdot AC^2 + n^2 \cdot BC^2 = (m+n)^2 OC^2.$$

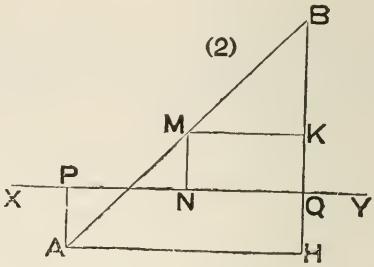
Cor. If AB is divided into $m - n$ equal parts; and if BA is produced to O, so that AO contains n, and BO m of these parts, preceding Theorem becomes

$$m \cdot AC^2 - n \cdot BC^2 = (m - n) \cdot CO^2 + m \cdot AO^2 - n \cdot BO^2,$$

which again may be modified as in *Note* (2).

THEOREM (18)—*The distance of the mid point of a finite line from a line of indefinite length, is half the sum of the distances of its extremities, when the lines do not intersect, and half the difference, when they do.*





Let M be the mid pt. of a finite line AB; XY any indefinite line, which either does not cut AB, fig. (1), or does cut AB fig. (2).

Draw AP, BQ, MN $\perp^s$ on XY;

and MK, AH $\perp^s$ on BQ (produced if necessary).

Then K is mid pt. of BH.

$$\therefore \quad KQ = \tfrac{1}{2}(BQ + HQ) \text{ in fig. (1).}$$

$$\therefore \quad MN = \tfrac{1}{2}(BQ + AP) \quad ,,$$

$$\text{And} \quad KQ = \tfrac{1}{2}(BQ \sim HQ) \text{ in fig. (2).}$$

$$\therefore \quad MN = \tfrac{1}{2}(BQ \sim AP) \quad ,,$$

Note—If in the above we consider that perpendiculars drawn from A and B in opposite directions towards XY are to be taken as additive in the one direction, and subtractive in the other, both above are included in

$$MN = \tfrac{1}{2}(AP + BQ).$$

With this convention 2 MN is said to be the *algebraic sum* of AP and BQ.

Many of the preceding and analogous propositions are particular cases of general theorems relating to a point of much geometrical importance, called the mean centre. Of this point a formal definition will now be given.

Def. Let there be any system of points A, B, C, D, &c., and a corresponding system of whole numbers a, b, c, d, &c.; let the line AB be divided into a + b equal parts, and let AM_1 contain b of them and BM_1 a of them; also let CM_1 be divided into a + b + c equal parts, and let M_1 M_2 contain c of them, and CM_2 a + b of them; and let this process be repeated so that a series of points M_3 M_4 &c., are similarly found; then if M is the last point found, it is called

the centre of mean position, or the mean centre of the points A, B, C, D, &c., *for the system of multiples* a, b, c, d, &c.

N.B.—The most important particular case is that in which

$$a = b = c = d = \&c.$$

For the satisfactory treatment of the properties of the mean centre it is essential that the preceding convention be adhered to—that if a set of perpendiculars are drawn to a line from points on opposite sides of it, all the perpendiculars on one side of the line are to be considered *additive*, and all on the other side *subtractive*. The idea of this convention is not to be found in Euclid; but is the very basis of Modern Geometry.

In connection with it we shall occasionally use the following very useful symbol.

Σ is to be read, and is solely the equivalent of—*the algebraic sum of all such quantities as.*

Examples—If we have a series of points A, B, C, &c., joined two and two, then—

(1) Σ (AB) means—*the sum of all the joins:*

(2) Σ ($\triangle$ ABC) means—*the sum of all the triangles which can be got by taking the joins three together:*

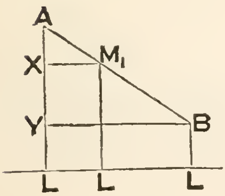
(3) Σ (AB^2) means—*the sum of all the squares on the joins.*

THEOREM (19)—*If* A, B, C, *&c., are a number of points,* a, b, c, *&c., corresponding whole numbers,* M *the* mean centre *of the points for those numbers; then, if perpendiculars* AL, BL, CL, *&c., and* ML, *are drawn to any line—every point in the line being denoted by* L—*the position of* M *is given by the relation—*

$$(a + b + c + \&c.) \,.\, ML = a \,.\, AL + b \,.\, BL + c \,.\, CL + \&c.$$

Or, which is the same thing briefly expressed,

$$ML \,.\, \Sigma (a) = \Sigma (a \,.\, AL).$$



1°, take the case of two pts. A, B.

Draw M_1X, BY $\perp$ to AL.

Then $a \,.\, AL = a \,.\, (M_1L + AX)$.

And $b \,.\, BL = b \,.\, (M_1L - XY)$.

$$\therefore\ a \,.\, AL + b \,.\, BL = (a + b) \,.\, M_1L + a \,.\, AX - b \,.\, XY.$$

Now since M_1X is $\parallel$ to BY

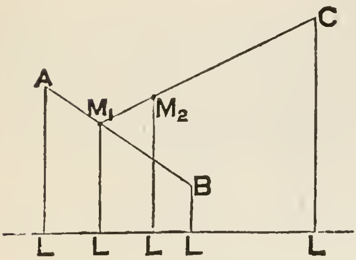
$\therefore$ M_1X divides AY into $a + b$ equal parts,

of which AX contains b parts, and XY contains a.

$$\therefore \quad a \,.\, AX \text{ and } b \,.\, XY \text{ each contain } ab \text{ parts.}$$

$$\therefore \quad a \,.\, AX - b \,.\, XY = 0,$$

$$\therefore \quad a \,.\, AL + b \,.\, BL = (a + b) \,.\, M_1L.$$



2°, taking a third pt. C; by what has been proved,

$$(a + b + c) \,.\, M_2L = (a + b) \,.\, M_1L + c \,.\, CL,$$

$$= a \,.\, AL + b \,.\, BL + c \,.\, CL.$$

And, by an obvious extension of the process, we get the required result.

Cor. (1) For any line which passes through the mean centre $ML = 0$, and

$$\therefore \quad a \,.\, AL + b \,.\, BL + c \,.\, CL + \&c. = 0;$$

or, briefly, when M lies in L,

$$\Sigma (a \,.\, AL) = 0.$$

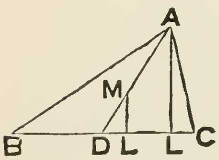
Cor. (2) In the case when $a = b = c = \&c.$

$$ML = \frac{1}{n}(AL + BL + \&c.)$$

where n = the number of points.

Some noticeable examples are—

(1) when there are three points whose joins form a $\triangle$; and $a = b = c$.



Take BC as the line L.

Then $ML = \frac{1}{3} AL$.

And, if AM meet BC in D, by drawing $\|^s$ to BC through M, and the mid pt. of AM, we get

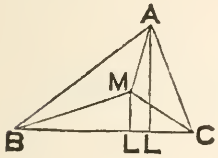
$$MD = \tfrac{1}{3} AD.$$

Similarly if BM meet AC in E, and we take AC as L, we should get

$$ME = \tfrac{1}{3} BE.$$

$\therefore$ M is the *centroid* of the $\triangle$.

(2) When there are three points whose joins form a Δ; and of such lengths that BC, CA, AB can be respectively divided into a, b, c parts, all of which are equal.



Take BC as L.

Then (BC + CA + AB) ML = BC . AL.

Now if r = dist. of the in-centre from each of the sides of Δ ABC,

$$r\,(BC + CA + AB) = 2 \text{ area of } \Delta\ ABC.$$

$$\text{Also } BC \,.\, AL = 2 \text{ area of } \Delta\ ABC.$$

$$\therefore\quad ML = r.$$

Similarly the distance of M from AC $= r$.

$\therefore$ M is the *in-centre.*

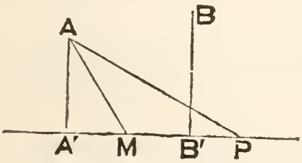
(3) when there are four points whose joins form a quad.; and a = b = c = d. It will be easily seen that M is the intersection of the joins of the mid pts. of opposite sides.

Note—If A, B, C, &c., are the positions, and a, b, c, &c., the respective numbers of units of mass of a system of material particles, the *mean centre* is the position of the *centre of mass* of the particles.

THEOREM (20)—*If* M *is the mean centre of the points* A, B, C, *&c., for the corresponding numbers* a, b, c, *&c.; and if* P *is any other point, then*—

$$\Sigma\,(a \,.\, AP^2) = \Sigma\,(a \,.\, AM^2) + PM^2 \,.\, \Sigma\,(a),$$

where n = *the number of points.*



Draw $\perp^s$ AA′, BB′, CC′, &c., from the pts. to the join of MP, produced indefinitely.

$$\text{Then } a \,.\, AP^2 = a \,.\, (AM^2 + PM^2 + 2\,A'M \,.\, MP).$$

And similar results hold for each of the pts.

Adding corresponding sides of all the results, we get

$$\Sigma\,(a \,.\, AP^2) = \Sigma\,(a \,.\, AM^2) + PM^2\,\Sigma\,(a) + 2\,PM\,\Sigma\,(a \,.\, A'M).$$

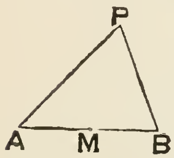
But, since M is the mean centre, it follows from Theorem (19) Cor. (1) that $\Sigma(a \,.\, A'M) = 0$; for if a $\perp$ is drawn thro. M to PA′, and if α, β, γ, &c. are the projections of A, B, C, &c. on that $\perp$, then $\Sigma(a \,.\, A\alpha) = 0$, and $\therefore \Sigma(a \,.\, A'M) = 0$

$\therefore$ required result follows.

Cor. (1). If $a = b = c = d =$ &c.

$$\Sigma(AP^2) = \Sigma(AM^2) + n \,.\, PM^2.$$

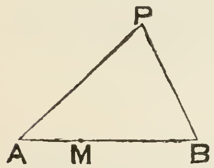
Cor. (2). In the particular case when there are two points, and $a = b$, we have



$$\begin{aligned} AP^2 + BP^2 &= AM^2 + BM^2 + 2\,PM^2, \\ &= 2\,AM^2 + 2\,PM^2. \end{aligned}$$

A result which has been proved independently as Theorem (11).

Cor. (3). When there are two points, and $a = 2$, $b = 1$, we have



$$\begin{aligned} 2\,AP^2 + BP^2 &= 2\,AM^2 + BM^2 + 3\,PM^2, \\ &= 6\,AM^2 + 3\,PM^2. \end{aligned}$$

This was proved as Theorem (15).

Cor. (4). When there are three points, and $a = b = c$, we get Theorem (16).

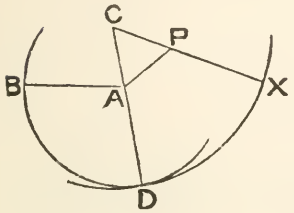
Note—The Theorem holds when the points A, B, C, &c., are collinear, and P is any other point collinear with them: then, by giving particular values to a, b, c, &c., various theorems will follow.

Theorem (17) is the general case for two points; and also holds when C is on AB; so that then, by giving various values to m and n, we can get different relations between the segments of a line.

APPENDIX.

CONTAINING PREVIOUSLY OMITTED PROOFS.

Book i. Proposition 2.



Let P be the given pt., and AB the given st. line.

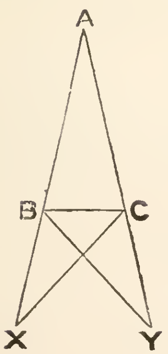
Join AP; and on AP describe the equilat. △ APC.

Let ⊙ with A as centre, and AB as radius, cut CA produced in D; and let ⊙ with C as centre, and CD as radius, cut CP produced in X.

Then CX = CD, being radii of same ⊙;
and CP = CA, being sides of an equilat. △.
∴ PX = AD.
But AD = AB, being radii of same ⊙.
∴ PX = AB.

i.e. from the given pt. P a line PX has been drawn equal to the given line AB.

Book i. Proposition 5.



Let ABC be a △ in which AB = AC.

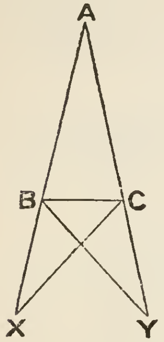
In the production of AB take any pt. X; and in the production of AC take Y, so that AY = AX.

Join CX, BY.

Then in △s ACX, ABY, since

$$\left.\begin{aligned} AC &= AB, \\ AX &= AY, \\ \text{and } C\hat{A}X &= B\hat{A}Y; \end{aligned}\right\}$$

∴ △ ACX ≡ △ ABY - - - (a)



Now $\because$ AX = AY, and AB = AC,

$\therefore$ BX = CY.

And as in the $\triangle^s$ BXC, CYB we have also

$$\left.\begin{aligned} CX &= BY, \\ \text{and } B\hat{X}C &= C\hat{Y}B; \end{aligned}\right\} \text{, from } (\alpha)$$

$$\therefore \triangle BXC \equiv \triangle CYB \quad - \; - \; - \; - \quad (\beta)$$

Now $\because$ $A\hat{C}X = A\hat{B}Y$, from (α)

and $B\hat{C}X = C\hat{B}Y$, from (β)

$\therefore$, subtracting corresponding sides, we have

$$A\hat{C}B = A\hat{B}C,$$

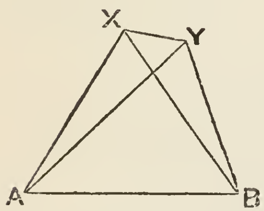
Also $C\hat{B}X = B\hat{C}Y$, from (β)

Book i. Proposition 7.

For suppose that on the same side of the same base AB, there are 2 $\triangle^s$ AXB, AYB,

$$\left.\begin{aligned} \text{such that } AX &= AY, \\ \text{and } BX &= BY. \end{aligned}\right\}$$

Join XY.



1°, let the vertex of each $\triangle$ be outside the other $\triangle$.

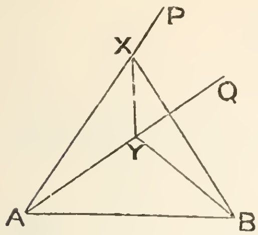
Then $A\hat{X}Y = A\hat{Y}X$, $\because$ AX = AY,

and $\therefore$ $< B\hat{Y}X$,

$\therefore$, also, $< B\hat{X}Y$, $\because$ BX = BY,

i.e. $< A\hat{X}Y$.

But this is an absurdity.



2°, let the vertex Y of one $\triangle$ lie within the other $\triangle$.

Produce AX to P, and AY to Q.

Then $\widehat{PXY} = \widehat{QYX}$, $\because$ AX = AY,

and $\therefore < \widehat{BYX}$,

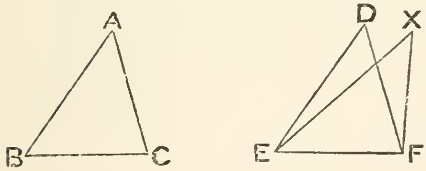
$\therefore$, also, $< \widehat{BXY}$, $\because$ BX = BY,

i.e. $< \widehat{PXY}$.

But this is an absurdity.

$\therefore$ the assumption that AX = AY, and BX = BY, *simultaneously*, leads to an absurdity; and $\therefore$ is not true:
i.e. there cannot be 2 such $\triangle$s as are assumed.

Book i. Proposition 8.



Let ABC, DEF be $\triangle$s, such that

$$\left.\begin{aligned} AB &= DE, \\ AC &= DF, \\ \text{and } BC &= EF. \end{aligned}\right\}$$

Suppose $\triangle$ ABC so applied to $\triangle$ DEF that
pt. B is on pt. E, and direction of BC on that of EF.
then pt. C will coincide with pt. F,
$\because$ BC = EF.

And if A, instead of falling on D, had a different position (as X) then on the same side of the same base EF there would be 2 $\triangle$s DEF, XEF,

$$\text{such that } \left.\begin{aligned} ED &= EX, \\ \text{and } FD &= FX. \end{aligned}\right\}$$

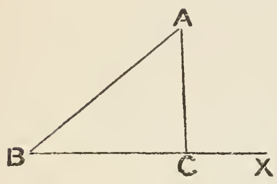
But this cannot be.

$\therefore$ A will coincide with D.

So that $\triangle$ ABC will coincide entirely with $\triangle$ DEF.

$\therefore \triangle ABC \equiv \triangle DEF$.

Book i. Proposition 17.



In any $\triangle$ ABC produce a side BC to X.

Then $\widehat{ACX} > \widehat{ABC}$.

$\therefore \widehat{ACX} + \widehat{ACB} > \widehat{ABC} + \widehat{ACB}$,

But $\widehat{ACX} + \widehat{ACB} = 2$ rt. $\wedge^s$.

$\therefore \widehat{ABC} + \widehat{ACB} < 2$ rt. $\wedge^s$.

Book ii. Proposition 8.

Let st. line AB be divided in X.

Produce AB to Y, so that BY = BX.

Then sq. on AX + 4 rect. under AB, BX,

= sq. on AX + 4 sq. on BX + 4 rect. under BX, AX (by ii. 3)

= sq. on AX + sq. on XY + 2 rect. under AX, XY ($\because$ BX = BY)

= sq. on AY (by ii. 4).

EXERCISES ON BOOK ii.

NOTE—*These Exercises are all Theorems to be proved; and depend mainly on the principles of Book* ii.

1. In a right-angled triangle, the sum of the hypotenuse and the perpendicular from the vertex of the right angle on the hypotenuse, is greater than the sum of the sides containing the right angle.

NOTE—*Take the sqs. on the two sums.*

2. If any point within a rectangle is joined to its corners, the sum of the squares on the joins to a pair of opposite corners is equal to the sum of the squares on the other pair of joins.

NOTE—*Join the point to the intersection of the diagonals, and use* ii. *Addenda* (11).

3. If AXB, AYB are right-angled triangles on same side of same hypotenuse AB; and AP, BQ are perpendiculars on XY produced, then

$$XP^2 + XQ^2 = YP^2 + YQ^2.$$

4. If the hypotenuse AB of a right-angled triangle ABC is trisected in X, Y; then

$$CX^2 + CY^2 + XY^2 = \tfrac{2}{3} AB^2.$$

NOTE—*Use* ii. *Addenda* (11), *and Cor.* ii. 4 (*a*).

5. If ABC is an isosceles triangle, and XY is parallel to the base BC; then, if BY is joined,

$$BY^2 = CY^2 + BC \,.\, XY.$$

NOTE—*From* X, Y *draw* $\perp^s$ *to* BC.

6. Any rectangle is equal to half the rectangle under the diagonals of the squares on two of its adjacent sides.

7. If from any point perpendiculars are dropped on all the sides of any rectilineal figure, the sum of the squares on the alternate segments are equal.

8. If ABC is any triangle, X a point in BC such that

$$AB^2 + BX^2 = AC^2 + CX^2,$$

and M the mid point of AX; then BM = CM.

9. If AB is the diameter of a circle; X, Y points in AB equidistant from the centre; P any point on the circumference; then

$$PX^2 + PY^2 = AX^2 + AY^2 = BX^2 + BY^2.$$

NOTE—*Use* ii. *Addenda* (11), *and* ii. 10.

10. If ABCD is a parallelogram such that BA = BD; then

$$BD^2 + 2\,BC^2 = AC^2.$$

11. If a line is divided in *medial section*, the sum of the squares on the whole line and on the lesser part is equal to three times the square on the other part.

12. If the square on an altitude of a triangle is equal to the rectangle under the segments into which it divides the base, then the vertical angle is right.

13. Conversely to the last exercise—If a triangle is right-angled, then the square on the altitude from the corner of the right angle is equal to the rectangle under the segments into which it divides the hypotenuse.

14. ABC is an isosceles triangle, whose vertex is A: if CX is the perpendicular on AB; and XP the perpendicular on BC; then

$$AB^2 = PA^2 + PX^2.$$

15. In any triangle ABC, if BP, CQ are perpendiculars on AC, AB (produced if necessary) then

$$BC^2 = AB \,.\, BQ + AC \,.\, CP.$$

16. If the extremities of any chord of a circle are joined to any point in the diameter parallel to the chord; then the sum of the squares on the joins is equal to the sum of the squares on the segments of the diameter.

17. If ABCD is a square, and X a point in AC such that $AX = \frac{1}{4}\,AC$; then

$$\text{figure } ABXD = 2\,AX^2.$$

18. If BX, CY are squares on sides BA, CA of any triangle ABC, then

$$BC^2 + XY^2 = 2\,(AB^2 + AC^2).$$

Note—*Draw* AN $\perp$ *to* XY; *and let* NA *meet* BC *in* M: *draw* BP, CQ $\perp$ *to* NAM.

19. If squares are described on three sides of any triangle, and their corners joined; then the sum of the squares on the hexagon thus formed is equal to four times the sum of the squares on the sides of the triangle.

Note—*Use preceding Exercise.*

20. The sum of the squares on the diagonals of any quadrilateral is equal to twice the squares on the joins of the mid points of opposite sides.

Note—*Use* i. *Addenda* (18), *and Cor.* ii. 4, (*a*).

21. If two sides of a quadrilateral are parallel, then the sum of the squares on its diagonals is equal to the sum of the squares on its non-parallel sides together with twice the rectangle under its parallel sides.

22. If A, B, C, D are four collinear points; X the mid point of AB, Y the mid point of CD, and M the mid point of XY; and if P is *any* point; then

$$PA^2 + PB^2 + PC^2 + PD^2 = MA^2 + MB^2 + MC^2 + MD^2 + 4PM^2.$$

NOTE—*Use* ii. 9, and ii. *Addenda* (11). *Or deduce from property of* mean centre on p. 113.

23. ABCD is any quadrilateral; the mid points of its diagonals are joined, and M is the mid point of this join; if P is *any* point; then

$$PA^2 + PB^2 + PC^2 + PD^2 = MA^2 + MB^2 + MC^2 + MD^2 + 4PM^2.$$

24. In the figure of ii. 11; if DX meets BE in Y, and (when produced) meets BF in Z; then DZ is perpendicular to BF and to AY.

25. If X, Y, Z are the feet of any concurrent perpendiculars on the sides, and D, E, F the mid points of the sides, respectively opposite corners A, B, C of a triangle; then, of the rectangles under BC, XD, under CA, YE, and under AB, ZF, the greatest is equal to the sum of the other two.

NOTE—*Use* ii. 13, *and* ii. *Addenda* (11).

26. If ABCD is a rectangle, X any point in BC, and Y any point in CD; then ABCD = 2 △ AXY + BX . DY.

27. If two opposite sides of a quadrilateral are bisected, then the sum of the squares on the other sides together with the squares on the diagonals is equal to the sum of the squares on the bisected sides together with four times the square on the join of their mid points.

28. In any triangle ABC, X, Y, Z are the feet of the altitudes, and O the orthocentre; then

$$AB^2 + BC^2 + CA^2 = 2\,(AX\,.\,AO + BY\,.\,BO + CZ\,.\,CO).$$

NOTE—*Use* ii. 12.

29. If a point is taken within a triangle at which its three sides subtend equal angles; then the sum of the squares on the sides of the triangle is equal to twice the sum of the squares on the joins of its corners to that point, together with the sum of the rectangles under these joins taken two and two.

NOTE—*Draw a* ⊥ *from one of the corners on one of the joins.*

30. In any quadrilateral the sum of the squares on the four lines from the middle of the join of the mid points of a pair of opposite sides to the corners of the quadrilateral, is equal to the sum of the squares on the joins of the mid points of opposite sides and the join of the mid points of the diagonals.

31. If p_1, p_2, p_3, p_4 are the successive perpendiculars from the corners of a square on any line (cutting, or not cutting it) then

$$p_1^2 + p_3^2 - 2p_2p_4 = \text{area square} = p_2^2 + p_4^2 - 2p_1p_3$$

BOOK iii.

Def. A straight line terminated at both ends by the circumference of a circle is called a **chord** of the circle.

Def. Every chord through the centre is called a **diameter**.

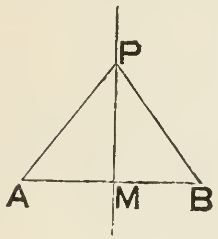
Def. Any part of the circumference of a circle is called an **arc** of the circle.

Def. A proposition which is proved as directly subsidiary to another proposition is called a **lemma**.

Proposition 1.

PROBLEM—*To find the centre of a given circle.*

Lemma—Any pt. P equidistant from two given pts. A and B, must lie in the $\perp$ to their join through its mid pt.



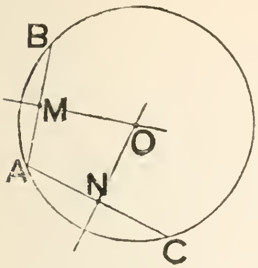
For, joining P to M the mid pt. of AB, in $\triangle$s PMA, PMB we have

$$\left.\begin{array}{r} PA = PB, \\ PM \text{ common}, \\ \text{and } AM = BM; \end{array}\right\}$$

$$\therefore \quad A\hat{M}P = B\hat{M}P;$$

and $\therefore$, as they are adjacent, each is right:

i.e. P lies in the $\perp$ to AB through M.



Now take AB, AC two chds. of the given ⊙.

Then the centre of the ⊙, being equidistant from A and B, must (by the *Lemma*) lie in the ⊥ to AB through its mid pt. M.

Similarly it must lie in the ⊥ to AC through its mid pt. N.

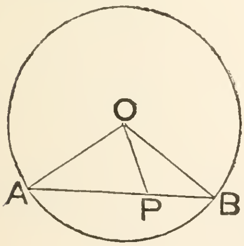
But these ⊥s will intersect in some pt. O,

$\because$ they make acute $\wedge$s with the join of M, N.

$\therefore$ O is the centre of the ⊙.

Proposition 2.

THEOREM—*If two points are taken on the circumference of a circle, the chord which joins them must lie within the circle.*



Let A, B be the pts.; O the centre of the ⊙.

Take P any pt. in AB.

Join OA, OB, OP.

Then $O\hat{A}B = O\hat{B}A$,

$\because$ $OA = OB$.

But $A\hat{P}O > O\hat{B}P$,

$\therefore$ also $> O\hat{A}P$,

$\therefore$ $OA > OP$:

i. e. the dist. of P from the centre < the radius.

$\therefore$ P lies within the ⊙.

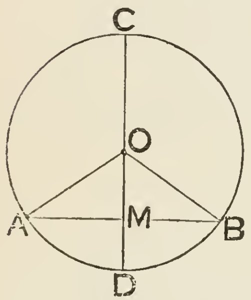
Similarly every other pt. in AB lies within the ⊙.

Proposition 3.

THEOREM—*If a diameter of a circle cuts a chord which is not a diameter, then if the chord is—*

(α) *bisected, it is also cut at right angles—*

(β) *cut at right angles, it is also bisected.*



Let CD a diam. of a ⊙, centre O, cut AB a chd., not a diam., in M.

Join OA, OB.

(α) let M be mid pt. of AB.

Then in $\triangle^s$ OAM, OBM, we have

$$\left.\begin{aligned} OA &= OB, \\ OM &\text{ common,} \\ \text{and } AM &= BM; \end{aligned}\right\}$$

∴ $\wedge^s$ at M are equal;

and ∴, being adjacent, are right:

i.e. CD is ⊥ to AB.

(β) let $\wedge^s$ at M be right.

Then in $\triangle^s$ OAM, OBM, we have

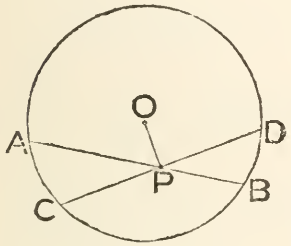
$$\left.\begin{aligned} OA &= OB, \\ \therefore \text{ also } O\hat{A}M &= O\hat{B}M, \\ \text{and } O\hat{M}A &= O\hat{M}B; \end{aligned}\right\}$$

∴ AM = BM:

i.e. AB is bisected in M.

Proposition 4.

THEOREM—*If two chords of a circle, which are not both diameters, cut one another, their point of intersection cannot be the middle point of both, though it may be the middle point of either.*



If one of the chds. is a diam. its mid pt. is the centre.

$\therefore$ it cannot be bisected by the other which, not being a diam., does not go through the centre.

Let these chds. AB, CD, neither of which is a diam., cut in P.

Join P with centre O.

Then, if OP bisects AB, $O\hat{P}B$ is a rt. $\wedge$.

And, if OP bisects CD, $O\hat{P}D$ is a rt. $\wedge$.

But these cannot happen simultaneously,

for then $O\hat{P}B$ would equal $O\hat{P}D$, a part of itself.

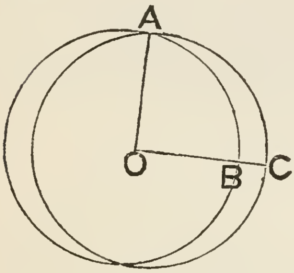
$\therefore$ P is not the mid pt. of *both* lines.

NOTE—The converse—*If a point is the mid point of two chords of a circle, these chords are diameters*—can be proved in an exactly similar manner.

Def. If one point is the centre of two or more circles, these circles are called **concentric**.

Proposition 5.

THEOREM—*If two circles cut one another they are not concentric.*



Let A be a pt. where two ⊙s cut.

Assume a common centre O.

Draw a line OBC cutting ⊙s in B, C.

Join OA.

Then OB = OA, being radii of same ⊙.

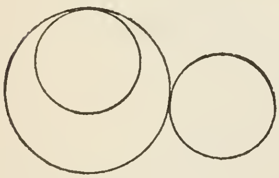
And OC = OA, " "

∴ OC = OB, a part of itself.

∴ the assumption of a common centre leads to an impossibility; and ∴ is not true :

i. e. the ⊙s are not concentric.

Def. Two circles are said to be **in contact** (or to **touch**) at a point, when they meet at that point without cutting each other.



Ax. If two circles touch, one must be wholly within, or wholly without the other.

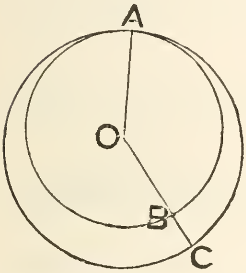
Def. When two circles are in contact, so that one is wholly within the other, the one within is said to have **internal** contact with the one without.

Def. When two circles are in contact, so that one is wholly without the other, the circles are said to have **external** contact.

Note—Any number of circles may have contact (either external or internal) at the same point.

Proposition 6.

THEOREM—*If one circle has internal contact with another, the circles cannot be concentric.*



Let A be a pt. where one ⊙ has internal contact with another.

Assume a common centre O.

Draw a line OBC cutting the ⊙s in B and C.

Join OA.

Then OB = OA, being radii of the same ⊙.

And OC = OA, " "

∴ OC = OB, a part of itself.

∴ the assumption of a common centre has led to an impossibility; and ∴ is not true:

i. e. the ⊙s are not concentric.

Note (1)—The *converses* of Props. 5 and 6 will be found in the *Addenda*.

Note (2)—Props. 5 and 6 may be included in this single enunciation—*If the circumferences of two circles meet at a point, they cannot be concentric.* The case when the circles meet by *external contact* is axiomatically true.

Proposition 7.

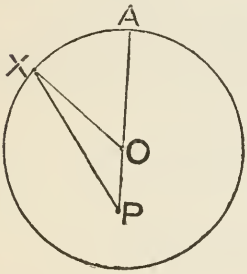
THEOREM—*If from a point (not the centre) within a circle, straight lines are drawn to the circumference—*

(α) *the greatest is that which goes through the centre ;*

(β) *the least is that which would, if it was produced, go through the centre ;*

(γ) *of any other two, that one is the greater which subtends the greater angle at the centre ;*

(δ) *any one of the lines (excepting the greatest and least) will have one other of the lines equal to it; but not more than one.*



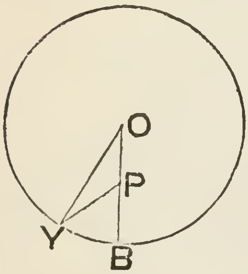
Let P be a pt. within a ⊙, whose centre is O.

Of lines drawn to the circumference from P—

(*a*) let POA be the one through O, and PX any other.

Join OX.

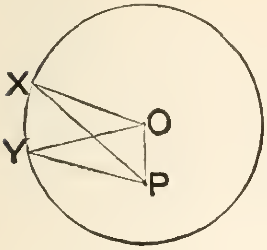
Then PA = PO + OA,
= PO + OX,
and ∴ > PX.



(β) let BP be the one which, when produced, goes through O.

Take PY any other; and join OY.

Then OY < PY + PO.
But PB + PO = OB = OY.
∴ PB + PO < PY + PO.
∴ PB < PY.

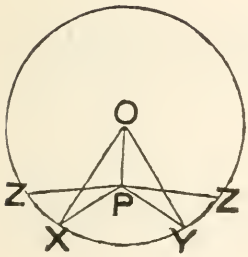


(γ) let PX, PY be any two *not* through O, such that $P\hat{O}X > P\hat{O}Y$.

Then in $\triangle^s$ OXP, OYP, since

$$\left.\begin{aligned} OX &= OY, \\ OP &\text{ is common,} \\ \text{but } P\hat{O}X &> P\hat{O}Y; \end{aligned}\right\}$$

$$\therefore \quad PX > PY.$$



(δ) let PX be any line, not the greatest or least.

Join OX; and draw another radius OY, so that $P\hat{O}Y = P\hat{O}X$.

Join PY.

Then in $\triangle^s$ PXO, PYO, we have

$$\left.\begin{aligned} OX &= OY, \\ OP &\text{ common,} \\ \text{and } P\hat{O}X &= P\hat{O}Y; \end{aligned}\right\}$$

$$\therefore \quad PX = PY.$$

Nor can any third line (as PZ) be drawn so that

$$PZ = PX = PY.$$

For however it is drawn it will subtend a different angle at O from PX and PY.

And $\therefore$ (γ) will be greater or less than PX and PY.

Note—It is customary in part (γ) of Prop. 7, as well as in the corresponding part of Prop. 8, to say—*that which is nearer the line through the centre is greater than the one more remote*—It is difficult to see how, of three *concurrent* lines, one can be properly said to be *nearer* to another than the third, without some arbitrary definition of the sense in which the word '*nearer*' is used—as, for example, that it is to mean—*more nearly coincident with.* To avoid this difficulty the wording of the enunciation has been changed.

Proposition 8.

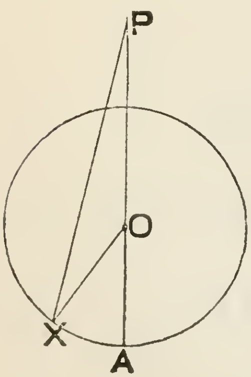
THEOREM—*If from a point outside a circle straight lines are drawn to meet its circumference—*

(α) *the greatest is that which goes through the centre;*

(β) *the least is that which would, if it was produced, go through the centre;*

(γ) *of any two which are incident on the concavity of the circumference, or of any two which are incident on the convexity, that one is the greater which subtends the greater angle at the centre;*

(δ) *any one of the lines, excepting the greatest and least, will have one other line equal to it, but not more than one.*

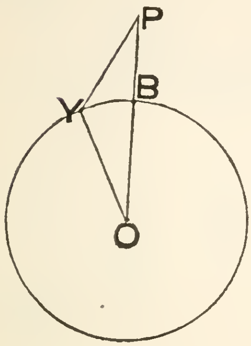


Let P be a pt. outside a ⊙, whose centre is O.

Of lines drawn to the circumference from P—

(α) let POA be the one through O, and PX any other. Join OX.

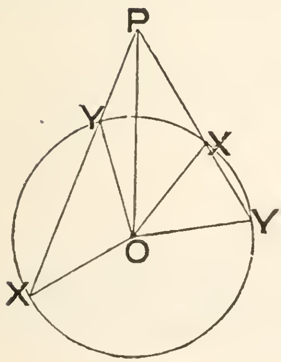
$$\text{Then } PA = PO + OA,$$
$$= PO + OX,$$
$$\text{and } \therefore \; > PX.$$



(β) let PB be that line which, when produced, goes through O.

Take PY any other line. Join OY.

$$\text{Then } PO < PY + YO,$$
$$\text{i.e. } PB + BO < PY + YO,$$
$$\text{or } PB < PY.$$

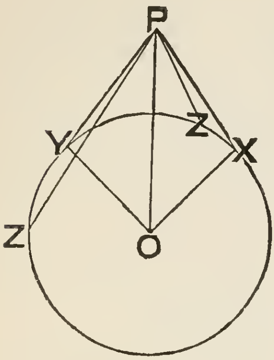


(γ) let PX, PY be any two lines, not through O, either *both* meeting the convexity, or *both* meeting the concavity; and such that

$$P\hat{O}X > P\hat{O}Y.$$

Then, in either case, since in $\triangle$s POX, POY, we have

$$\left.\begin{array}{r} OX = OY, \\ OP \text{ common}, \\ \text{but } P\hat{O}X > P\hat{O}Y; \end{array}\right\}$$
$$\therefore \; PX > PY.$$



(δ) let PX be any line, not the greatest or least.

Join OX, and draw another radius OY, so that $P\hat{O}Y = P\hat{O}X$.

Join PY.

Then in $\triangle$s POX, POY, we have

$$\left.\begin{aligned} OX &= OY, \\ PO &\text{ common,} \\ \text{and } P\hat{O}X &= P\hat{O}Y; \end{aligned}\right\}$$

$$\therefore \quad PX = PY.$$

Nor can any third line (as PZ) be drawn so that

$$PZ = PX = PY.$$

For however it is drawn, it will subtend a different $\wedge$ at O from PX or PY.

And $\therefore$ by (γ) will be greater or less than PX and PY.

Proposition 9.

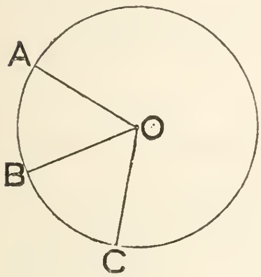
THEOREM—*If more than two equal lines can be drawn from a point within a circle to the circumference, that point is the centre.*

For if one line is drawn from a point, within a circle but not the centre, to the circumf., only one other can be drawn from the point so that the two may be equal.

$\therefore$ if three equal lines can be drawn from a point to circumf. that point must be the centre.

Proposition 10.

THEOREM—*Two circles (which do not coincide) cannot have more than two points in common.*



Let O be the centre of a ⊙.

Then if A, B, C are three pts. on its circumf., they are equidistant from O.

∴ if A, B, C could be on circumf. of another ⊙, O would be the centre of this other ⊙ ;

and then two concentric ⊙s would cut :

which cannot be.

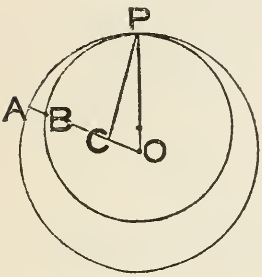
∴, unless the ⊙s coincide, they cannot have three pts. in common.

Note—The arrangement of the enunciation and proof, given above, avoids the awkwardness of endeavouring to draw the impossible figure of two concentric circles, cutting in more than two points.

Def. The straight line on which lie the centres of two or more circles is called their **line of centres.**

Proposition 11.

THEOREM—*If one circle has internal contact with another, their line of centres must go through a point of contact.*



Let O be centre of outer $\odot$, and C centre of inner $\odot$.

Then if OC produced do not go through a pt. of cont., it must cut the inner $\odot$ *first,* say in B; and the outer $\odot$ *afterwards,* say in A.

Let P be a pt. of cont. Join OP, CP.

Then $OC + CP > OP$.

But $OP = OA$, being radii of same $\odot$.

$\therefore \quad OC + CP > OA$,

$\therefore$ *à fortiori* $> OB$,

i.e. $> OC + CB$.

$\therefore \quad CP > CB$.

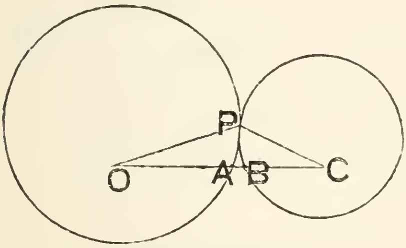
But $CP = CB$, being radii of same $\odot$.

$\therefore$ the assumption that OC does not go through a pt. of cont. leads to a contradiction; and $\therefore$ is not true:

i.e. OC goes through a pt. of cont.

Proposition 12.

THEOREM—*When two circles have external contact their line of centres must go through a point of contact.*



Let O and C be the centres of two ⊙s which have ext. contact in P.

Then if OC do not go through a pt. of cont. it must cut each ⊙ in a different pt.: suppose it cuts ⊙s in A and B respectively.

Join OP, CP.

Then OP + CP > OC,

∴ *à fortiori* > OA + CB.

But OP = OA, being radii of same ⊙.

And CP = CB, „ „

∴ OP + CP = OA + CB.

∴ the assumption that OC does not go through a pt. of cont. leads to a contradiction; and ∴ is not true:

i. e. OC goes through a pt. of cont.

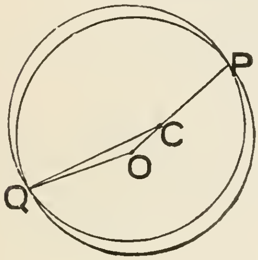
Note (1)—It is customary, in enunciating this proposition, and the preceding, to say that the line of centres passes through *the point of contact of the circles*: this assumes iii. 13.

Note (2)—The converses of Props. 11 and 12 will be found in the *Addenda*.

Proposition 13.

THEOREM—*Two circles cannot touch in more than one point.*

Let O, C be the centres of two ⊙s in contact: then OC, the line of centres, passes through *a* pt. of contact—say P.



Assume Q to be another pt. of cont.

Join OQ, CQ.

1°, let the contact be *internal.*

Then $OQ < OC + CQ$.

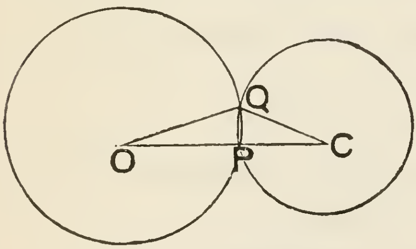
But $CQ = CP$, being radii of same ⊙.

$\therefore \quad OQ < OC + CP$,

i. e. $< OP$.

But $OQ = OP$, being radii of same ⊙.

$\therefore$ the assumption of a second pt. of *internal* contact leads to a contradiction; and $\therefore$ is not true.



2°, let the contact be *external.*

Then $OC < OQ + CQ$;

or $OP + CP < OQ + CQ$.

But $OP = OQ$, being radii of same ⊙.

And $CP = CQ$, „ „

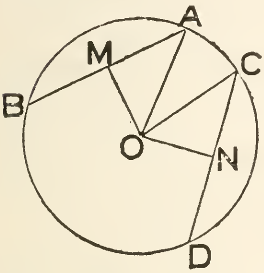
$\therefore OP + CP = OQ + CQ$.

$\therefore$ the assumption of a second pt. of *external* contact leads to a contradiction; and $\therefore$ is not true:

i. e. in neither case is there a second pt. of contact.

Proposition 14.

THEOREM—*In a circle chords which are—*
(α) *equal, must be equally distant from the centre;*
(β) *equally distant from the centre, must be equal.*



Lemma—If in 2 rt. angled $\triangle$s the hypotenuses are equal, and another pair of sides are equal, the remaining pair of sides must be equal. This is an immediate consequence of i. 47.
See p. 60, *Theorem* (8)

Let AB, CD be chds. of a $\odot$ whose centre is O. Join OA, OC.
Draw OM, ON respectively $\perp$ to AB, CD.
$\therefore$ M, N are mid. pts. of AB, CD.
(α) if AB = CD, so that AM = CN.
Then in $\triangle$s OMA, ONC, we have

$$\left.\begin{aligned} OA &= OC, \\ AM &= CN, \\ \text{and } \angle\text{s at } M &\text{ and } N \text{ right;} \end{aligned}\right\}$$

$\therefore$, by the *Lemma*, OM = ON:
i. e. AB, CD are equidistant from centre.
(β) if OM = ON.
Then, in $\triangle$s OMA, ONC, we have

$$\left.\begin{aligned} OA &= OC, \\ OM &= ON, \\ \text{and } \angle\text{s at } M &\text{ and } N \text{ right;} \end{aligned}\right\}$$

$\therefore$, by the *Lemma*, AM = CN.
$\therefore$ AB = CD.

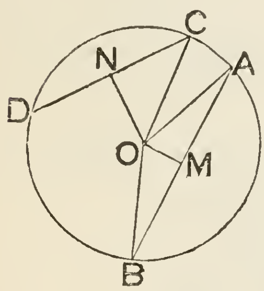
Proposition 15.

THEOREM—*In a circle, of all chords which can be drawn—*

(α) *the diameter is the greatest;*

(β) *that one, of any two, which is nearer the centre, is greater than the other;*

(γ) *the greater, of any two, is nearer the centre than the other.*



Let AB be any chd., not a diam., of a ⊙ whose centre is O.

Join OA, OB.

Then (α) the diam. of ⊙, being equal to the sum of two radii,

$$= OA + OB,$$

and $\therefore > AB$.

Next, let CD be any other chd.

Draw OM, ON $\perp^s$ on AB, CD respectively. Join OC.

Then sq. on AM + sq. on OM = sq. on OA,

= sq. on OC,

= sq. on CN + sq. on ON.

$\therefore$ (β), if OM < ON, so that sq. on OM < sq. on ON,

sq. on AM > sq. on CN;

or AM > CN:

i. e. AB > CD.

And (γ), if AB > CD,

then AM > CN, so that sq. on AM > sq. on CN,

sq. on OM < sq. on ON.

$\therefore$ OM < ON.

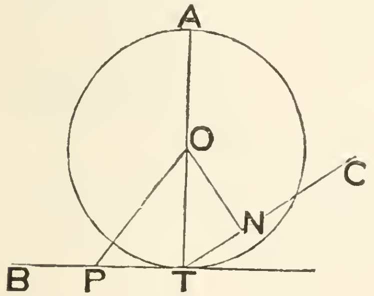
Def. A straight line is said to **touch**, or be a **tangent** to a circle, if the line meets the circle, but does not, when produced, cut it.

Proposition 16.

THEOREM—*In a circle—*

(*a*) *a straight line perpendicular to a diameter at one of its extremities touches the circle ;*

(β) *any other line through the same extremity cuts the circle.*



Let O be the centre of a ⊙ ; and AOT a diam.

(*a*) At the pt. T let TB, ⊥ to AT, be drawn; and let P be any other pt. in this ⊥.

Join OP.

Then, since $O\hat{T}P$ is right, $O\hat{P}T$ is acute.

∴ OP > OT, the radius of the ⊙.

∴ P lies without the ⊙.

Similarly every other pt. in TB, except T, can be proved to lie outside the ⊙.

∴ PT touches the ⊙.

(β) through T draw any other line TC, making with AT, $A\hat{T}C$ acute.

Draw ON ⊥ to TC.

Then ON < OT, a radius of the ⊙.

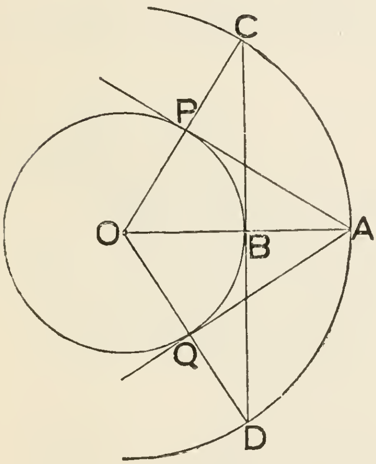
∴ N lies within the ⊙.

∴ TC cuts the ⊙.

Note—The preceding Prop. gives the solution of the Problem—*To draw a tangent to a circle at a given point on its circumference.*

Proposition 17.

PROBLEM—*From a point outside a circle to draw a tangent to the circle.*



Let A be the given pt. outside a ⊙ whose centre is O.

Join OA, cutting the ⊙ in B.

With centre O and radius OA describe a ⊙.

Draw a line through B ⊥ to OA meeting the outer ⊙ in C and D.

Join OC, OD cutting the given ⊙ in P, Q, respectively.

Join AP, AQ.

Then in $\triangle^s$ OPA, OBC, we have

$$\left.\begin{array}{r} OP = OB, \\ OA = OC, \\ \text{and } \hat{O} \text{ common}; \end{array}\right\}$$

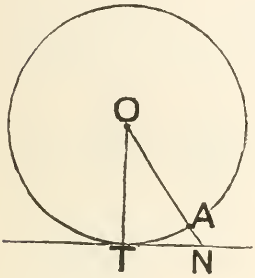
$$\therefore \quad O\hat{P}A = O\hat{B}C = \text{a rt. } \wedge.$$

$\therefore$ AP is a tangent to given ⊙.

Similarly AQ „ „

Proposition 18.

THEOREM—*If a straight line touch a circle, the radius drawn to the point of contact must be perpendicular to the touching line.*



Let a ⊙, whose centre is O, be touched at T by a st. line.

Then if the ⊥ ON from O on that tang. does not go through T, it must cut the ⊙ in some pt. A. Join OT.

Then ∵ $\widehat{ONT}$ is right, $\widehat{OTN}$ is acute.

∴ ON < OT, a radius.

But ON > OA, another radius.

∴ the assumption that the ⊥ from O to the tang. at T does not go through T leads to a contradiction; and ∴ is not true.

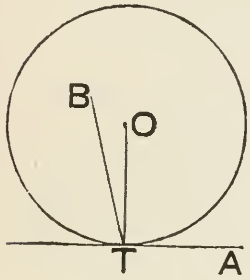
i.e. OT is ⊥ to the tang. at T.

Def. The straight line perpendicular to a tangent, at its point of contact with a circle, is called a **normal** to the circle.

Note—The preceding Prop. is therefore equivalent to—*All radii of a circle are normals to it.* Similarly the following Prop. is equivalent to—*All normals to a circle will go through its centre.*

Proposition 19.

THEOREM—*If a straight line is a tangent to a circle the perpendicular to it at the point of contact must go through the centre.*



Let TA be a tang. at T to a ⊙ whose centre is O; and let TB be ⊥ to TA.

Then OT is a radius drawn to T the pt. of cont. of a tang.

∴ OT is ⊥ to TA.

But BT is ⊥ to TA.

∴ $A\hat{T}B = A\hat{T}O$.

But of these ∧s one is a part of the other, *unless* TB go through O.

∴ TB must go through O.

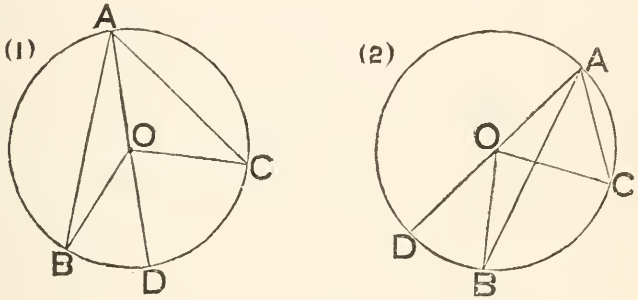
Def. The figure which is formed by a chord of a circle, and either of the arcs which it cuts off, is called a **segment** of the circle.

Def. An angle contained by two straight lines drawn from a point in the arc of a segment to the extremities of its chord is called an **angle in the segment.**

Def. The angle formed by any two chords drawn from a point on the circumference of a circle is called **an angle at the circumference**; and is said to *stand* on the arc intercepted between the chords.

Proposition 20.

THEOREM—*The angle at the centre of a circle is double the angle at the circumference, standing on the same arc.*



Let BC be an arc of a ⊙, on which stand $\widehat{BOC}$ at centre O, and $\widehat{BAC}$ at circumf.

Join AO; and produce it to meet circumf. in D.

Then $\widehat{OAB} = \widehat{OBA}$, $\because$ OA = OB.

$\therefore$ $\widehat{BOD}$ (which = sum of these $\wedge$s) = 2 $\widehat{OAB}$.

Similarly $\widehat{COD} = 2\,\widehat{CAO}$.

In fig. (1), where O is within $\widehat{BAC}$, by adding corresponding sides of these equals, we get

$$\widehat{COD} + \widehat{BOD} = 2\,(\widehat{CAO} + \widehat{BAO})$$

$$\text{i.e. } \widehat{COB} = 2\,\widehat{CAB}.$$

In fig. (2), where O is without $\widehat{BAC}$, by taking the difference of the corresponding sides, we get

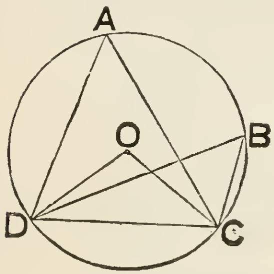
$$\widehat{COD} \sim \widehat{BOD} = 2\,(\widehat{CAO} \sim \widehat{BAO})$$

$$\text{i.e. } \widehat{COB} = 2\,\widehat{CAB}.$$

The case when O is on BA or CA is involved in the earlier part of the proof.

Proposition 21.

THEOREM—*The angles in the same segment of a circle are equal.*



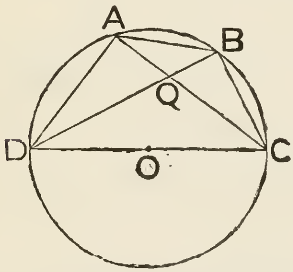
Let O be the centre of a ⊙; CD the chd. of a segt. of it.

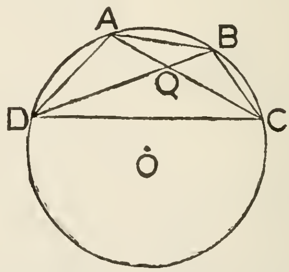
Take A, B any two pts. in the arc of the segt.; and join OC, OD, AC, AD, BC, BD.

1°, when O lies on the same side of DC that A and B do, so that the segt. > semi ⊙,

$$D\hat{A}C \text{ and } D\hat{B}C \text{ are each half } D\hat{O}C.$$

$$\therefore \quad D\hat{A}C = D\hat{B}C.$$





2°, when O is either on CD, or on the side of CD remote from A and B, so that each segt. =, or < semi ⊙.

Join AB; and let Q be the intersection of a pair of the lines DB, CA, forming the $\wedge^s$.

$$\text{Then } D\hat{A}Q + A\hat{D}Q = D\hat{Q}C = Q\hat{B}C + B\hat{C}Q.$$

$$\text{But } A\hat{D}Q = B\hat{C}Q, \text{ by the 1st case.}$$

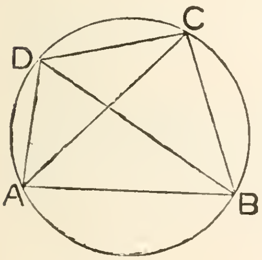
$$\therefore \quad D\hat{A}C = D\hat{B}C.$$

Def. When any number of points are on the same circle they are said to be **concyclic.**

Def. When four points are concyclic, the quadrilateral formed by their joins is called a **cyclic quadrilateral.**

Proposition 22.

THEOREM—*The opposite angles of a cyclic quadrilateral are supplementary.*



Let ABCD be a cyclic quad.

Join AC, BD.

Then $B\hat{A}C = B\hat{D}C$, in same segt.

And $C\hat{A}D = C\hat{B}D$, „ „

∴, adding corresponding sides of these equals, we get

$$B\hat{A}D = B\hat{D}C + C\hat{B}D.$$

Add $B\hat{C}D$ to each side, then

$$B\hat{A}D + B\hat{C}D = \text{the three } \angle^s \text{ of } \triangle\, BCD,$$

$$\text{i.e.} = \text{two rt. } \angle^s.$$

∴ $B\hat{A}D$ and $B\hat{C}D$ are supplementary.

Similarly it can be proved that

$C\hat{D}A$ and $C\hat{B}A$ are supplementary.

Note—The converses of Props. 21 and 22 will be found in the *Addenda.*

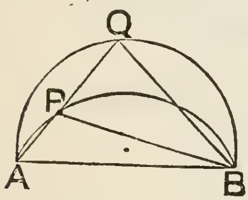
Def. A line which cuts a circle is called a **secant.**

Note—It has been already noticed that a secant will cut a circle in two, and only two, points—see i. *Addenda* (2) *Cor.* (2).

Def. If the angle in a segment of one circle is equal to the angle in a segment of another circle, the segments are called **similar.**

Proposition 23.

THEOREM—*Two similar segments of circles on the same side of a common chord must coincide.*



Assume that on the same side of AB, as a common chd., there are two similar segts. not coinciding.

Since the segts. cut in A, B, they cannot cut again.

$\therefore$ one segt. lies wholly within the other.

Draw secant APQ to cut the inner segt in P, and the outer segt. in Q; and join BP, BQ.

Then $A\hat{P}B = A\hat{Q}B$, since segts. are similar.

But ext. $A\hat{P}B$, of $\triangle$ BPQ, $>$ int. opposite $A\hat{Q}B$.

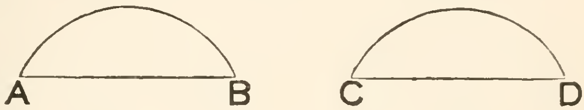
$\therefore$ the assumption has led to a contradiction;

and $\therefore$ is not true:

i. e. if similar segts. are placed on the same side of AB they will coincide.

Proposition 24.

THEOREM—*Similar segments of circles on equal chords are identically equal.*



Let there be similar segts. on the equal chds. AB, CD.

Apply them to each other so that—

A may be on C, and direction of AB on that of CD.

Then B will coincide with D,

$\because$ AB = CD.

And the segts. will coincide,

$\because$ they are similar, and on same side of a common chd.

i. e. segt. on AB $\equiv$ segt. on CD.

Proposition 25.

PROBLEM—*Given an arc of a circle to draw the rest of the circle.*

Same solution as Prop. 1 of this Book.

Ax. If a circle is superposed on an equal circle, so as to coincide with it wholly—

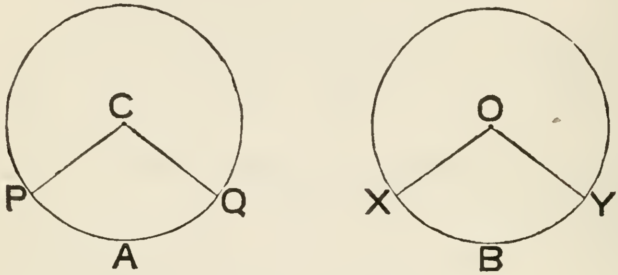
1°, any arcs of the circles which coincide are equal:

2°, if an arc falls on an equal arc, so that one pair of their extremities coincide, the other pair of their extremities will coincide.

3°, if an arc falls on an arc, so that their pairs of extremities coincide, the arcs coincide throughout.

Proposition 26.

THEOREM—*In equal circles the arcs which subtend equal angles, whether at the centres or circumferences, are equal.*



Let C and O be the centres of equal ⊙s; and let PAQ, XBY be arcs in them such that $P\hat{C}Q = X\hat{O}Y$.

It will be sufficient to prove the Prop. for the case of central ∧s, ∵ if the ∧s at the circumfs. are equal, those at the centre, being double of them, must also be equal.

Superpose one ⊙ on the other, so that

C may be on O,
CP on OX,
and arc PA along arc XB.

Then CQ will fall on OY,

∵ $P\hat{C}Q = X\hat{O}Y$.

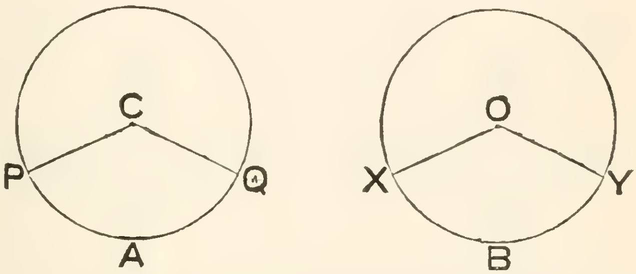
∴ Q will coincide with Y.

∴ arcs PAQ, XBY coincide.

∴ arc PAQ = arc XBY.

Proposition 27.

THEOREM—*In equal circles the angles which stand upon equal arcs are equal, whether they are at the centres or circumferences.*



Let C and O be the centres of equal ⊙s; and PAQ, XBY equal arcs in them respectively, subtending $P\hat{C}Q$ and $X\hat{O}Y$.

It will be sufficient to prove the central ∧s equal, ∵ those at the circumfs. are halves of them.

Superpose one ⊙ on the other, so that

C may be on O,
CP on OX,
and arc PA along arc XB.

Then Q will coincide with Y,

∵ arc PAQ = arc XBY.

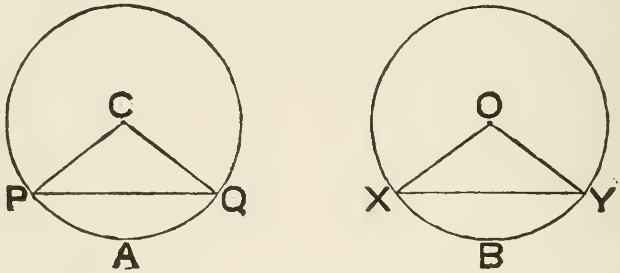
∴ CQ will coincide with OY.

∴ $P\hat{C}Q = X\hat{O}Y$.

Def. When a chord of a circle divides the circumference unequally the greater arc is called the **major arc**, and the lesser arc is called the **minor arc**.

Proposition 28.

THEOREM—*If in equal circles equal chords are drawn so as to divide each circumference unequally, then the two major arcs are equal, and the two minor arcs are equal.*



Since a major and its corresponding minor arc make up the circumf., it is sufficient to prove the Prop. for the minor arcs.

Let C and O be the centres of equal ⊙s; and PQ, XY equal chds. in them respecty. which cut off the minor arcs PAQ, XBY.

Join CP, CQ, OX, OY.

Then in the △s CPQ, OXY, we have

CP = OX, being radii of equal ⊙s,
CQ = OY, " "
and PQ = XY;

∴ $P\hat{C}Q = X\hat{O}Y$.

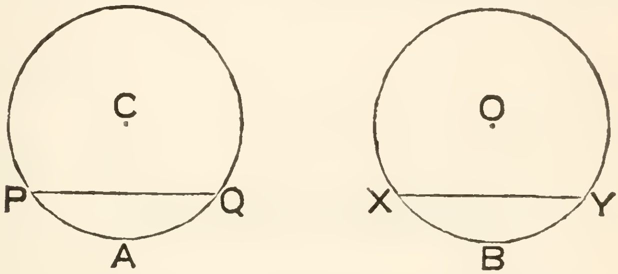
If ∴ one ⊙ is superposed on the other, so that

C may be on O,
CP on OX,
and arc PAQ along arc XBY;

then CQ will fall on OY;
and Q will coincide with Y,
∴ arcs PAQ, XBY coincide.
∴ arc PAQ = arc XBY.

Proposition 29.

THEOREM—*If in equal circles chords are drawn so as to cut off equal arcs, these chords are equal.*



Let C and O be the centres of equal $\odot^s$, in which equal arcs PAQ, XBY are respectively cut off by the chds. PQ, XY.

Superpose one $\odot$ on the other, so that

C may be on O,
P on X,
and PAQ along XBY.

Then Q will coincide with Y,

$\because$ arc PAQ = arc XBY.

$\therefore$ PQ = XY.

Note—The four preceding Props. are not usually proved by *superposition*; nor are the proofs here given, by that method, shorter than the usual proofs. The great advantage of using it here is that each Prop. is thus made independent of the rest, so that it is not necessary to recollect the order of the propositions. It is to be noted that in Prop. 28, when it is proved (in line 9 from the end) that $P\hat{C}Q = X\hat{O}Y$, the conclusion follows at once from Prop. 26. The rest is put in to obviate the need to recollect the order.

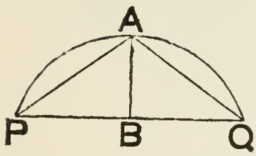
Note—It is obvious that the four preceding Props. are true when, for the words *equal circles,* is substituted, *the same circle:* the superposition can then be effected by turning the circle about its centre as a pivot; one of the arcs must then be supposed to remain fixed while the other is brought over it.

Ax. Every diameter of a circle bisects its circumference.

Note—It is an immediate inference from this axiom that—the arc of any segment of a semi-circle is a *minor* arc.

Proposition 30.

PROBLEM—*To bisect a given arc of a circle.*



Let PAQ be the given arc.

Join PQ; and bisect it in B.

Draw BA, $\perp$ to PQ, to meet the arc in A. Join PA, QA.

Then in $\triangle$s PBA, QBA, we have

$$\left.\begin{array}{r} PB = QB, \\ AB \text{ common}, \\ \text{and } P\hat{B}A = Q\hat{B}A; \end{array}\right\}$$

$$\therefore \quad AP = AQ.$$

$\therefore$ minor arc cut off by AP = minor arc cut off by AQ.

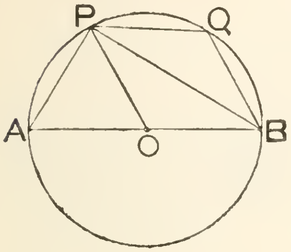
But the parts into which arc PAQ is divided at A are minor arcs.

$\because$ AB produced is a diam.

$\therefore$ BA bisects given arc PAQ.

Proposition 31.

THEOREM—*In a circle the angle in a segment which is—*
(α) *a semicircle, is a right angle ;*
(β) *greater than a semicircle, is less than a right angle ;*
(γ) *less than a semicircle, is greater than a right angle.*



Let O be the centre of a ⊙ ; AOB a diam. ; P any pt. in one of the semi ⊙s formed by AB.

Join PA, PB, PO.

(α) since OP = OA, being radii,

$\therefore \; O\hat{P}A = O\hat{A}P.$

$\therefore \; P\hat{O}B$ (which = their sum) $= 2\, O\hat{P}A.$

Similarly $P\hat{O}A = 2\, O\hat{P}B.$

$\therefore \; 2\,(O\hat{P}A + O\hat{P}B) = P\hat{O}B + P\hat{O}A,$

= two rt. ∧s.

$\therefore \; A\hat{P}B$ is a rt. ∧.

(β) $\therefore$ also $P\hat{A}B < A\hat{P}B,$

i.e. < a rt. ∧.

And it is the ∧ in segt. PAB, which > semi ⊙.

(γ) take Q any pt. in minor arc PB ; and join QP, QB.

Then $P\hat{A}B + P\hat{Q}B$ = two rt. ∧s,

$\because$ ABQP is a cyclic quad.

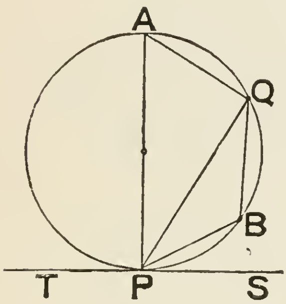
And it has been shown that $P\hat{A}B$ < rt. ∧.

$\therefore \; P\hat{Q}B$ > a rt. ∧.

And it is the ∧ in segt. PQB, which < semi ⊙.

Proposition 32.

THEOREM—*If a straight line touch a circle, and from the point of contact a straight line is drawn dividing the circle into two segments, the angles made by this line with the tangent are equal to the angles which are in the alternate segments.*



Let P be a pt. on the circumf. of a ⊙, at which TPS is drawn to touch it; and PQ to divide it into two segts.

Draw the diam. PA; and in the segt. not contg. A take any pt. B.

Join AQ, QB, BP.

Then $A\hat{Q}P$, in a semi ⊙, is right.

$\therefore \quad \hat{A} + A\hat{P}Q =$ a rt. $\wedge$,

$= S\hat{P}Q + A\hat{P}Q.$

$\therefore \quad S\hat{P}Q = \hat{A}$, the $\wedge$ in segt. alternate to SPQ.

Again, since APBQ is a cyclic quad.

$\therefore \quad \hat{A} + \hat{B} =$ two rt. $\wedge$s,

$= S\hat{P}Q + T\hat{P}Q.$

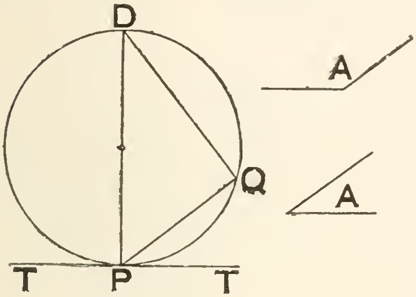
And it has been shown that $S\hat{P}Q = \hat{A}$.

$\therefore \quad T\hat{P}Q = \hat{B}$, the $\wedge$ in segt. alternate to TPQ.

Note—The *converse* of this follows at once, and is often useful.

Proposition 33.

PROBLEM—*On a given straight line to describe a segment of a circle containing an angle equal to a given angle.*



If the given $\wedge$ is right, a semi ⊙ on the given line will be the segt. required.

If not, let $\hat{A}$ be given $\wedge$; and PQ given st. line.

Make $Q\hat{P}T$ equal to $\hat{A}$.

Draw PD, QD respectively ⊥ to PT, PQ; and let them meet in D.

Describe the ⊙ of which PD is diam.

This ⊙ will go thro. Q, ∵ $P\hat{Q}D$ is rt.

Then PT, being ⊥ to diam. PD, touches ⊙.

∴ $Q\hat{P}T = \wedge$ in altern. segt.

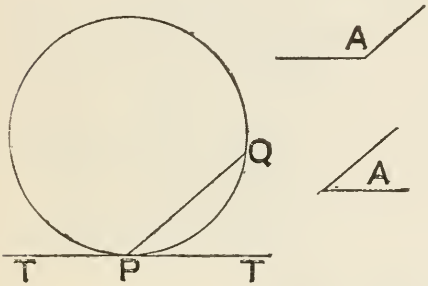
When A is acute this will be the segt. in which D lies; but when A is obtuse it will be the segt. in which D does not lie.

∴ on PQ a segt. has been described contg. an $\wedge$ equal to $\hat{A}$.

Note—The preceding Prop. might be enunciated thus—*On a given straight line to describe a segment of a circle, similar to a given segment.*

Proposition 34.

PROBLEM—*To cut off from a given circle a segment which shall contain an angle equal to a given angle.*



Let $\hat{A}$ be the given $\wedge$; and P a pt. on the given $\odot$.

Draw PT to touch the $\odot$ at P; and PQ a chd. of the $\odot$, such that

$$T\hat{P}Q = \hat{A}.$$

The two positions of T, in the fig., correspond to the cases of $\hat{A}$ acute, or obtuse.

Then $\because$ PT touches $\odot$, and PQ cuts it,

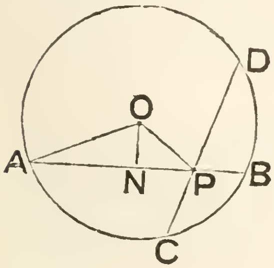
$\therefore$ $Q\hat{P}T = \wedge$ in altern. segt.

$\therefore$ segt. cut off by PQ contains an $\wedge$ equal to $\hat{A}$.

Note—The preceding Prop. might be enunciated thus—*From a given circle to cut off a segment similar to a given segment.*

Proposition 35.

THEOREM—*If two chords of a circle cut each other, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other.*



Let chds. AB, CD of ⊙, whose centre is O, cut in P.

Join OA, OP; and draw ON ⊥ to AB.

Since AB is divided equally in N, and unequally in P,

∴ rect. under AP, PB + sq. on PN = sq. on AN.

∴, adding sq. on ON to each side, and recollecting that

$$\left.\begin{array}{l}\text{sq. on PN + sq. on ON = sq. on OP,} \\ \text{and sq. on AN + sq. on ON = sq. on OA,}\end{array}\right\} \because \hat{N} \text{ is right,}$$

we get

rect. under AP, PB + sq. on OP = sq. on OA,

i. e. = sq. on a radius.

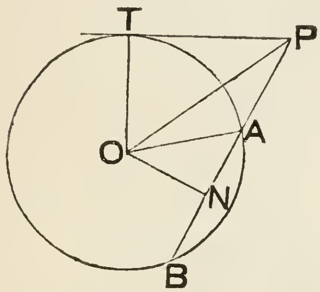
Similarly rect. under CP, PD + sq. on OP = sq. on a radius.

∴ rect. under AP, PB = rect. under CP, PD.

Note—The *converse* of Prop. 35 will be found in the *Addenda.*

Proposition 36.

THEOREM—*If from any point outside a circle, two straight lines are drawn, one of which cuts the circle and the other touches it; then the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, is equal to the square on the line which touches it.*



Let P be a pt. outside a ⊙, whose centre is O; and from P let PT be drawn to touch ⊙ in T, and PAB to cut it in A and B.

Join OA, OP, OT; and draw ON ⊥ to AB.

Since AB is bisected in N, and produced to P,

∴ rect. under PA, PB + sq. on AN = sq. on PN.

∴, adding sq. on ON to each side, and recollecting that

$$\left.\begin{array}{r} \text{sq. on AN} + \text{sq. on ON} = \text{sq. on OA}, \\ \text{and sq. on PN} + \text{sq. on ON} = \text{sq. on OP}, \end{array}\right\} \because \hat{\text{N}} \text{ is right,}$$

we get

rect. under PA, PB + sq. on OA = sq. on OP,

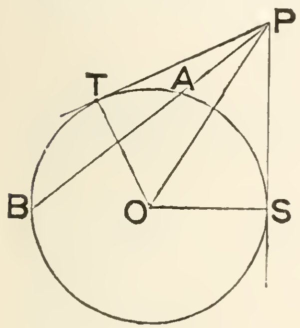
= sq. on PT + sq. on OT.

But sq. on OA = sq. on OT.

∴ rect. under PA, PB = sq. on PT.

Proposition 37.

THEOREM—*If from a point outside a circle, two straight lines are drawn, one to cut the circle, and the other to meet it; and if the square on the line which meets it, is equal to the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, then the line which meets the circle touches it.*



Let P be a pt. outside a ⊙, whose centre is O; and from P let PAB be drawn to cut the ⊙, and PT to meet it, so that

rect. under PA, PB = sq. on PT.

Join PO; and, on the side of PO remote from PT, draw PS to touch the ⊙. Join OT, OS.

Then since PS touches the ⊙, and PAB cuts it,

∴ sq. on PS = rect. under PA, PB,

= sq. on PT.

∴ PS = PT.

And since in $\triangle$s POT, POS, we have

$$\left.\begin{array}{r} PT = PS, \\ OT = OS, \\ \text{and } PO \text{ common}; \end{array}\right\}$$

∴ $P\hat{T}O = P\hat{S}O$ = a rt. ∧.

∴ PT touches the ⊙.

ADDENDA TO BOOK iii.

THE FOLLOWING ARE THE MOST OBVIOUS COROLLARIES TO THE PROPS. IN BOOK iii.

iii. 2. The whole circumference of a circle is concave to its centre.

iii. 3. (α) If a line cut any number of concentric circles, the intercepts made on it by any two of the circles are equal.

(β) If a series of parallel chords are drawn in a circle, all their mid points lie on the same diameter.

iii. 4. (α) If two chords of a circle bisect each other, they are diameters.

(β) If the corners of a parallelogram are concyclic, it is a rectangle.

iii. 5, and 6. Concentric unequal circumferences cannot meet either by intersection or contact.

iii. 10. Only one circle can go through three points.

Note—Hence the name *circumcentre* for the point equidistant from three points [see i. *Addenda* (20)]; for it is the centre of the only circle that goes through them, or circumscribes (see *Defs.* of iv.) the triangle formed by their joins.

iii. 11 and 12. The join of the centres of two touching circles is equal to the sum or difference of their radii, according as the contact is external or internal.

iii. 13. The join of the centres of two circles, which do not meet, is greater than the sum, or less than the difference of their radii, according as one circle lies without or within the other; and conversely.

iii. 16. (α) The line perpendicular to a diameter of a circle at its extremity, is the only tangent to the circle at that extremity.

(β) If any number of circles touch at a common point, they have a common tangent at that point.

iii. 17. The two tangents to a circle, from an external point are equal.

iii. 18. If a chord of a circle touch a lesser concentric circle, the point of contact is the mid point of the chord.

iii. 26. (α) Parallel chords of a circle intercept equal arcs; and conversely.

(β) Two perpendicular diameters quadrisect the circumference.

(γ) If the opposite angles of a cyclic quadrilateral are equal, the diagonal *not* passing through their vertices is a diameter.

(δ) Each pair of non-adjacent arcs, intercepted between perpendicular chords, together make a semi-circumference; and conversely.

Def.—If a trapezium has the angles adjacent to either of its parallel sides equal, it is called a **symmetrical trapezium**.

(ϵ) The corners of a symmetrical trapezium are concyclic; and, conversely, if a trapezium is cyclic it is symmetrical.

Note—A trapezium obviously is cyclic if—

1°, its parallel sides have a perpendicular bisector;

or, 2°, its diagonals are equal.

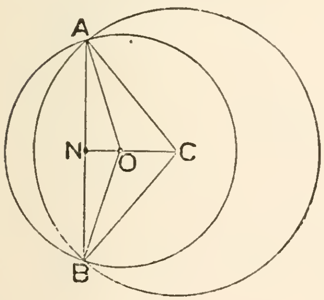
iii. 32. If any number of circles touch at the same point, either externally or internally, a line through their point of contact cuts off similar segments from all of them.

iii. 36. (α) If through a point, within or without a circle, two lines are drawn to meet the circumference, the rectangle under the segments of the one is equal to the rectangle under the segments of the other.

(β) Tangents to two intersecting circles, from any point in the production of their common chord, are equal.

THE FOLLOWING ARE SOME IMMEDIATE DEVELOPMENTS OF THE PROPS. IN BOOK iii.—NOT SO OBVIOUS AS TO BE PROPERLY CALLED COROLLARIES.

THEOREM (1)—*If two circles cut one another, their line of centres bisects their common chord.*



Let ⊙s, centres C and O, cut in A and B.

Let CO meet chd. AB in N.

Join CA, CB, OA, OB.

Then in △s CAO, CBO, we have

$$\left.\begin{aligned} CA &= CB, \\ OA &= OB, \\ \text{and } CO &\text{ common;} \end{aligned}\right\}$$

$$\therefore \widehat{ACO} = \widehat{BCO}.$$

Again in △s CAN, CBN, we have

$$\left.\begin{aligned} CA &= CB, \\ CN &\text{ common,} \\ \text{and } \widehat{ACN} &= \widehat{BCN}; \end{aligned}\right\}$$

$$\therefore AN = BN.$$

Note—The preceding Theorem virtually contains in itself the *converse* of iii. 11 and 12.

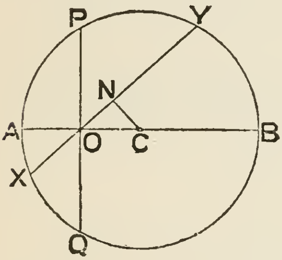
For, by iii. 10, two $\odot^s$ cannot cut in more than two points.

And, by the Theorem, if they cut in two points their line of centres cannot go through either of the points.

$\therefore$ if the line of centres go through a point in which two $\odot^s$ meet, the $\odot^s$ must touch at that point.

But this is the converse of iii. 11 and 12.

THEOREM (2)—*The shortest chord that can be drawn through a given point within a circle is the one which is perpendicular to the diameter through that point.*



Let O be given pt. within $\odot$, centre C;
AOCB the diam. through O;
POQ the chd. $\perp$ to AB;
XOY any other chd. through O;
CN $\perp$ to XY.

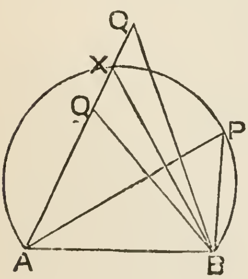
Then $\widehat{CNO}$ (being right) $>$ $\widehat{CON}$,

$\therefore$ CO $>$ CN,

$\therefore$ XY $>$ PQ:

i. e. PQ is the shortest chd. through O.

THEOREM (3)—(*Converse of* iii. 21.) *If any number of triangles, on the same base, and on the same side of it, have equal vertical angles, their vertices all lie on the arc of a segment of a circle, of which the base is a chord.*



Let ABP, ABQ be any two of Δ^s, on same side of same base AB, such that

$$\widehat{APB} = \widehat{AQB}.$$

Assume that $\odot$ through A, B, P cuts AQ (or AQ produced) in X; and join BX.

Then $\widehat{AQB} = \widehat{APB} = \widehat{AXB}$, in same segt.

But $\widehat{AQB} < \widehat{AXB}$, when X is in AQ;

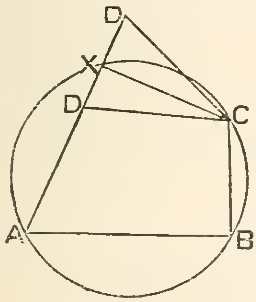
or $> \widehat{AXB}$, when X is in production of AQ,

∴ the assumption that Q is not on arc APB has led to a contradiction; and ∴ is not true:

i. e. Q is on arc APB.

Similarly every two, and ∴ all, of the vertices are on arc APB.

THEOREM (4)—(*Converse of* iii. 22.) *If the opposite angles of a quadrilateral are supplementary the quadrilateral is cyclic.*



Let ABCD be a quad. such that

$\widehat{B} + \widehat{D} =$ two rt. $\angle^s$.

Assume that ⊙ through A, B, C cuts AD (or AD produced) in X; and join CX.

Then $\widehat{ADC} =$ suppt. $\widehat{ABC}$,

$= \widehat{AXC}$, ∵ ABCX is cyclic.

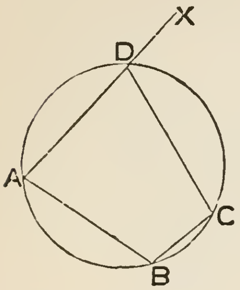
But $\widehat{ADC} < \widehat{AXC}$, when X is in AD;

or $> \widehat{AXC}$, when X is in production of AD.

∴ the assumption that D is not concylic with A, B, C has led to a contradiction; and ∴ is not true:

i. e. ABCD is a cyclic quad.

THEOREM (5)—*In a cyclic quadrilateral the external angle, made by producing a side at one of its corners, is equal to the angle of the quadrilateral at the opposite corner; and conversely.*



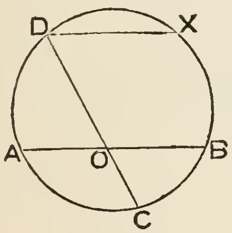
Let ABCD be a cyclic quad.; and let AD be produced to X.

Then $\widehat{CDX}$ = suppt. $\widehat{ADC}$,

= $\widehat{ABC}$.

And the *converse* (which is the more useful part of the Theorem) follows easily by an indirect proof, similar to that of Theorem (4).

THEOREM (6)—*If two chords of a circle intersect within it, the angle between them is equal to the circumferential angle on an arc which is equal to the sum of the arcs subtended by that angle between the chords which is under consideration.*



Let AOB, COD be chds. of a $\odot$, cutting at a pt. O within it.

Draw chd. DX ∥ to AB.

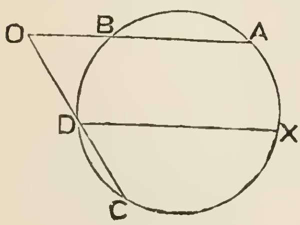
Then arc AD = arc BX;

∴ arc CBX = arc AD + arc CB.

And $\widehat{BOC}$ = $\widehat{CDX}$,

= $\wedge$ on arc CBX.

THEOREM (7)—*If two produced chords of a circle intersect without it, the angle between them is equal to the circumferential angle on an arc which is equal to the difference of the arcs intercepted between the chords.*



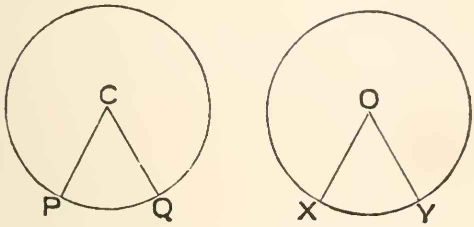
Let AB, CD be chds. of a $\odot$, which, being produced, meet at pt. O, outside it.

Draw chd. DX ∥ to AB.

Then arc BD = arc AX;
$\therefore$ arc CX = arc AC − arc BD.
And $\hat{AOC} = \hat{CDX}$,
$= \wedge$ on arc CX.

Def. A **sector** of a circle is the plane figure contained by two radii and the arc they intercept.

THEOREM (8)—*In equal circles (or the same circle) sectors on equal arcs are identically equal.*

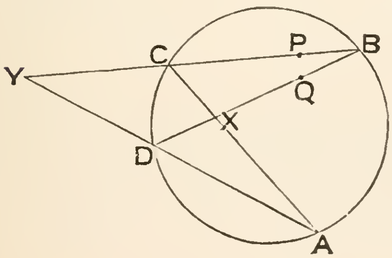


Let PCQ, XOY be sectors on equal arcs PQ, XY, of equal $\odot$s, whose centres are C, O.

$\therefore \hat{PCQ} = \hat{XOY}$

Superpose sector PCQ on sector XOY, so that
C may be on O,
and CP in direction OX.
Then CQ will be in direction OY,
$\because \hat{PCQ} = \hat{XOY}$.
Also P will coincide with X, and Q with Y,
$\therefore$ arc PQ will coincide with arc XY.
$\therefore$ sector PCQ $\equiv$ sector XOY.

THEOREM (9)—[*Converse of* iii. 35 *and Cor.* (a) *to* 36.] *If four points are so situated that the rectangle under the distances of two of them from the intersection of their joins (or joins produced) is equal to the rectangle under the distances of the other two from the same intersection, then the four points are concyclic.*



Let A, B, C, D, be four points such that joins AC, BD meet in X; and joins BC, AD produced, meet in Y.

1°, let XA . XC = XB . XD.

Assume that ⊙ through three of the pts, say C, D, A, meets BD in some point Q (not B).

Then XB . XD = XA . XC = XQ . XD.

∴ XB = XQ.

But this is absurd, for one of them is a part of the other.

2°, let YA . YD = YB . YC.

Assume that ⊙ through C, D, A meets CB in some pt. P (not B).

Then YB . YC = YA . YD = YP . YC.

∴ YB = YP.

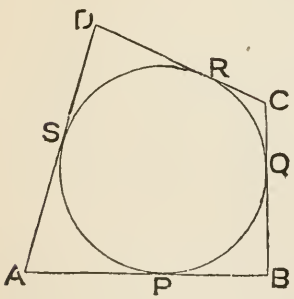
And again this is absurd, for one of them is a part of the other.

∴ in both cases the assumption that A, B, C, D are not concyclic leads to an absurdity; and ∴ is not true:

i. e. A, B, C, D are concyclic.

Some useful Theorems, mainly depending on Book iii.

Theorem (10)—*If each side of a quadrilateral touches the same circle, the sum of one pair of its opposite sides is equal to the sum of the other pair; and conversely.*



Let sides AB, BC, CD, DA, of quad. ABCD, touch a ⊙ at pts. P, Q, R, S, respectively.

Then AP = AS.

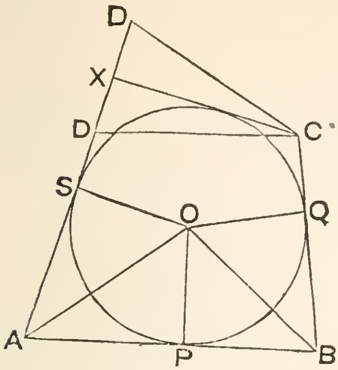
∵ they are tangents from same pt. to ⊙.

Similarly BP = BQ, CQ = CR, &c.

∴ AB + CD = AP + BP + CR + DR,

= AS + BQ + CQ + DS,

= AD + BC.



Next let AB + CD = AD + BC.

Bisect $\widehat{DAB}$ and $\widehat{CBA}$ by AO, BO; and from pt. O draw OP, OQ, OS respectively $\perp$ to AB, BC, DA.

Then in Δ^s OAS, OAP, we have

$$\left.\begin{array}{c} \widehat{OAS} = \widehat{OAP}, \\ \widehat{OSA} = \widehat{OPA}, \\ \text{and OA common;} \end{array}\right\}$$

$\therefore$ OS = OP, and similarly = OQ.

$\therefore$ $\odot$ with O as centre, and any one of them as radius, will go through P, Q, S; and will touch AB, BC, DA at those pts., $\because$ the $\wedge^s$ at P, Q, S are right.

Assume that this $\odot$ does not touch CD.

Draw CX a tangent to the $\odot$, so that it meets AD (or AD produced) in X.

Then AB + CD = BC + AD.

and AB + CX = BC + AX.

$\therefore$ CD $\sim$ CX = AD $\sim$ AX, i. e. = DX.

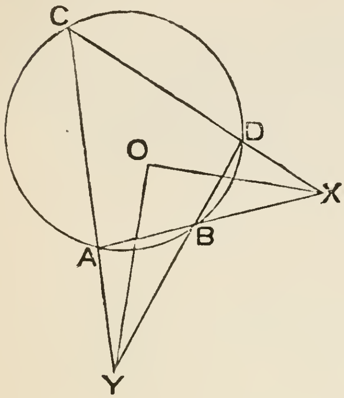
But CD $\sim$ CX $<$ DX.

$\therefore$ the assumption that $\odot$ does not touch CD leads to a contradiction; and $\therefore$ is not true:

i. e. sides of quad. all touch same $\odot$.

Note—From preceding, and iii. 22, we see that if the opposite $\wedge^s$ of a quad. are supplementary, and also the sum of one pair of opposite sides = the sum of the other pair, the quad. has its corners on one circle, and its sides touch another $\odot$: or (see *Defs.* of iv) it circumscribes one $\odot$, and is inscribed in another.

THEOREM (11)—*The bisectors of the angles formea by producing the opposite sides of a cyclic quadrilateral to meet, are at right angles.*

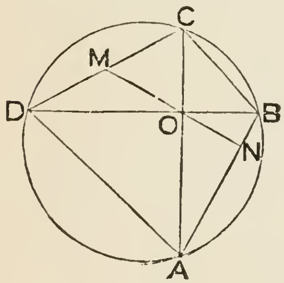


Let ABDC be a cyclic quad.; X the intersection of AB, CD, Y of DB, CA.

Let XO, YO be the bisectors of the $\wedge^s$ at X and Y: then

$$\widehat{XOY} = \widehat{XCY} + \widehat{CXO} + \widehat{CYO}$$
$$= \tfrac{1}{2}(2\,\widehat{XCY} + \widehat{CXB} + \widehat{CYB})$$
$$= \tfrac{1}{2}(\widehat{XCY} + \widehat{XBY})$$
$$= \text{a rt. } \wedge.$$

THEOREM (12)—(*Brahmegupta's*) *If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from their intersection on any side, being produced, bisects the opposite side; and, conversely, the median from their intersection bisecting a side, being produced, is perpendicular to the opposite side.*



Let ABCD be a cyclic quad. such that its diagonals AC, BD are $\perp$ at O.

Let NO, $\perp$ to AB, be produced to meet CD in M.

Then $\widehat{COM} = \widehat{AON} = \text{compt. } \widehat{BON} = \widehat{OBN} = \widehat{OCM}$;

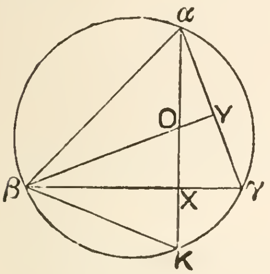
$\therefore$ MC = MO, and similarly = MD.

Again, let MO, bisecting CD, be produced to meet AB in N.

Then $\widehat{OAN} + \widehat{AON} = \widehat{ODC} + \widehat{COM} = \widehat{ODC} + \widehat{OCD} = \text{a rt. } \wedge.$

$\therefore$ $\widehat{ONA}$ is a rt. $\wedge$.

THEOREM (13)—*The foot of an altitude of a triangle is half way between the orthocentre and the point in which that altitude produced meets the circum-circle.*



Let alt. αX of $\triangle\, \alpha\beta\gamma$ meet circum-⊙ in K. Join βK; and draw alt. βY cutting αX in O, the orthocentre.

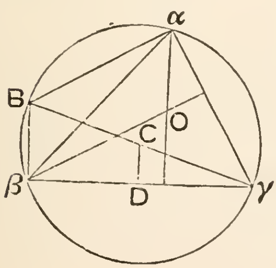
$$\text{Then } K\widehat{\beta}\gamma = K\widehat{\alpha}\gamma, \text{ in same segt.}$$
$$= \text{compt. } \alpha\widehat{\gamma}X,$$
$$= Y\widehat{\beta}\gamma.$$

Also, in $\triangle^s\ \beta XO$, βXK, we have

$$\left.\begin{aligned} \beta\widehat{X}O &= \beta\widehat{X}K, \\ \text{and } \beta X &\text{ common;} \end{aligned}\right\}$$
$$\therefore\quad OX = KX.$$

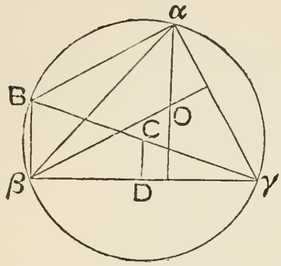
Cor. (1) $\alpha X \,.\, XO = \alpha X \,.\, XK = \beta X \,.\, X\gamma$.
Cor. (2) $\triangle\, \beta K\gamma \equiv \triangle\, \beta O\gamma$.

THEOREM (14)—*The join of any corner of a triangle to its orthocentre is double the distance of its circum-centre from the side opposite that corner.*



Let C be the circum-centre, O the orthocentre of any $\triangle\, \alpha\beta\gamma$.

Produce γC to meet circum-⊙ in B; and join $B\alpha$, $B\beta$.



Then $\because$ $\widehat{\gamma\beta B}$ is right, and C mid pt. of $B\gamma$,

$\therefore$ $B\beta$ is $\parallel$ to CD, and $=$ 2 CD.

Also $B\beta$, αO are both $\perp$ to $\beta\gamma$, and $\therefore$ $\parallel$.

Similarly, $B\alpha$, βO are both $\perp$ to $\alpha\gamma$, and $\therefore$ $\parallel$.

$\therefore$ $B\beta O\alpha$ is a $\square$.

$\therefore$ $\alpha O = B\beta = 2\,CD$.

And similarly for the other joins.

Cor. If α', O' are other positions of α, O; then $\alpha O = 2\,CD = \alpha' O'$;

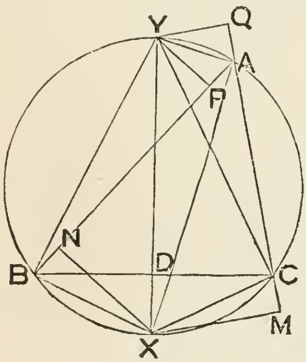
$\therefore$ OO' is $\parallel$ to, and $= \alpha\alpha'$.

THEOREM (15)—*The bisectors of the internal and external vertical angles of a triangle meet the circum-circle in points such that—*

(α) *they are the mid points of the arcs into which the base divides the circumference;*

(β) *their join is the diameter which bisects the base;*

(γ) *the feet of the perpendiculars from them on the sides are distant from the corners of the triangle by half the sum or half the difference of the sides.*



Let $\odot$ round $\triangle$ ABC meet the bisector of $\widehat{BAC}$ in X; and the bisector of $\widehat{BAQ}$, ext. to $\widehat{BAC}$, in Y.

Join XY, XB, XC, YB, YC.

Then since $\widehat{BAX} = \widehat{CAX}$,

$\therefore$ arc BX $=$ arc CX.

Again $\because$ YBCA is a cyclic quad., and CA produced to Q.

$\therefore$ $\widehat{YBC} = \widehat{YAQ} = \widehat{YAB} = \widehat{YCB}$, in same segt.

$\therefore$ arc YAC $=$ arc YB.

$\therefore$ (α) X, Y are mid pts. of arcs cut off by BC.

Let XY cut BC in D.

Then in $\triangle^s$ BDX, CDX, we have

$$\left\{\begin{array}{l} \widehat{DBX} = \widehat{DCX}, \text{ for they are on equal arcs,} \\ \widehat{DXB} = \widehat{DXC}, \text{ for same reason,} \\ \text{and } DX \text{ common;} \end{array}\right.$$

$$\therefore \quad BD = CD;$$

and $\wedge^s$ at D are right,

$\therefore$ (β) XY is diam. bisecting BC.

Lastly draw XN $\perp$ to AB, and XM $\perp$ to AC produced.

Then $\because$ CABX is a cyclic quad., and AC produced to M,

$$\therefore \quad \widehat{ABX} = \widehat{XCM}.$$

And in $\triangle^s$ XBN, XCM, we have also

$$\left\{\begin{array}{l} \widehat{XNB} = \widehat{XMC}, \\ \text{and } XB = XC, \text{ being on equal arcs;} \end{array}\right.$$

$$\therefore \quad BN = CM.$$

Also in $\triangle^s$ ANX, AMX, we have

$$\left.\begin{array}{l} \widehat{ANX} = \widehat{AMX}, \\ \widehat{NAX} = \widehat{MAX}, \\ \text{and } AX \text{ common;} \end{array}\right\}$$

$$\therefore \quad AN = AM.$$

Hence $AB - AN = AM \text{ (or } AN) - AC;$

$$\therefore \quad AN = \tfrac{1}{2}(AB + AC) = AM.$$

And $AC + CM \text{ (or } BN) = AB - BN;$

$$\therefore \quad BN = \tfrac{1}{2}(AB - AC) = CM.$$

Similarly if YP, YQ are corresponding $\perp^s$ from Y, it could be proved that

$$BP = \tfrac{1}{2}(AB + AC) = CQ,$$

and $AP = \tfrac{1}{2}(AB - AC) = AQ;$

$\therefore$ (γ) is true.

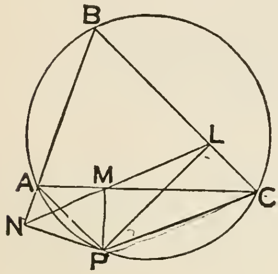
Cor. (1).

$$\widehat{BXN} \text{ (or } \widehat{CXM}) = \text{compt. } \widehat{XBN} = \widehat{ABY} \text{ (or } \widehat{ACY}) = \widehat{AXY} = \tfrac{1}{2}(\widehat{ABC} \sim \widehat{ACB}).$$

Cor. (2).

$$\widehat{YBD} \text{ (or } \widehat{YCD}) = \text{compt. } \widehat{XBD} = \widehat{BXD} \text{ (or } \widehat{CXD}) = \widehat{BAY} = \tfrac{1}{2}(\widehat{ABC} + \widehat{ACB}).$$

THEOREM (16)—(*Simson's*) *If from any point on the circumference of the circle through the three corners of a triangle, perpendiculars are dropped on its sides, produced when necessary, the feet of those perpendiculars are collinear; and conversely, if the feet of the perpendiculars from a point on the sides of a triangle are collinear, the point is concyclic with the corners of the triangle.*



Let P be any pt. on circumf. of ⊙ through the corners of △ ABC; PL, PM, PN ⊥s on sides opposite A, B, C respectively.

Join PA, PC, LM, MN.

Then $\widehat{PMN} = \widehat{PAN}$, ∵ PMAN is cyclic,

$= \widehat{BCP}$, ∵ PABC is cyclic,

= suppt. $\widehat{LMP}$, ∵ PCLM is cyclic.

∴ L, M, N are collinear.

Next let L, M, N be collinear.

Then $\widehat{CPL} = \widehat{CML} = \widehat{AMN} = \widehat{APN}$;

∴ $\widehat{CPA} = \widehat{NPL}$ = suppt. $\widehat{B}$;

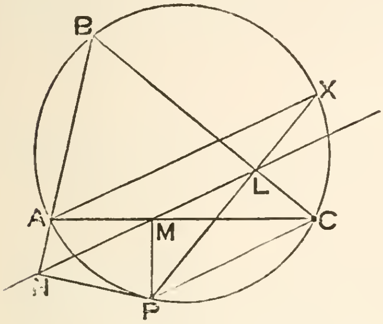
∴ C, P, A, B are concyclic.

Note—The line LMN, in preceding Theorem, is called **Simson's Line** (sometimes also **the pedal line**) for the triangle ABC, with respect to the point P.

Cor. (1). If PL, PM, PN are obliques, making equal ∧s (measured the same way round) with the sides; it may be shown similarly that L, M, N are collinear.

Cor. (2). If four △s are formed by four intersecting lines, the intersection of ⊙s about any two of them, is such that the feet of ⊥s from it on the four lines are collinear.

THEOREM (17)—*If, in the preceding Theorem,* PL, PM, PN, *are produced to meet the circumference in* X, Y, Z, *respectively, then* AX, BY, CZ *are parallel to Simson's Line.*



For $\widehat{AXP} = \widehat{ACP}$, in same segt.

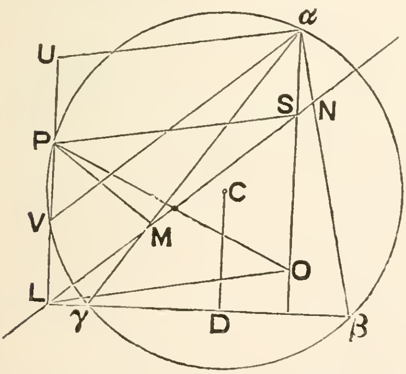
$= \widehat{PLM}$, $\because$ PMLC is cyclic.

$\therefore$ AX is $\parallel$ to LMN.

Simrly. for BY, CZ.

Cor. $\wedge$ between Simson's Lines of P, Q, $= \wedge$ subtended by PQ at circumf.

THEOREM (18)—*Simson's Line bisects the join of the orthocentre and that point on the circumference of the circum-circle with respect to which the Line is constructed.*



Let P be any pt. on $\odot$ round $\triangle\, \alpha\beta\gamma$; L, M, N the respective feet of $\perp$s from P on $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, so that LMN is Simson's Line.

Let PL meet $\odot$ again in V; and produce LP to U, so that PU = VL; join αU, αO, LO, where O is the orthocentre; from C, the centre of the $\odot$, draw CD $\perp$ to $\beta\gamma$; and join P to S, the pt. of intersection of αO and LMN.

Then the $\perp$ from C on PV bisects PV;

$\therefore$ UL = 2 CD = αO;

$\therefore$ ULOα is a ▱.

Also, αV being $\parallel$ to LMN, αVLS is a ▱.

$\therefore$ αS = VL = PU;

$\therefore$ αUPS is a ▱;

$\therefore$ PS is $\parallel$ to Uα, and $\therefore$ also to LO;

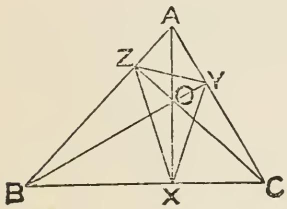
$\therefore$ PLOS is a ▱;

$\therefore$ PO is bisected by LS.

Cor. If $\alpha\beta\gamma$ is equilat. C and O coincide; and then Simson's Line bisects the radius of the circum-⊙ drawn to P.

Def. If the feet of the altitudes of a triangle are joined, the new triangle, formed by the joins, is called the **pedal triangle** with respect to the original triangle.

THEOREM (19)—*Each pair of sides of the pedal triangle makes equal angles with that altitude of the original triangle which is concurrent with them.*



Let ABC be any △; X, Y, Z the feet of the altitudes drawn respectively from the vertices A, B, C: then XYZ is the pedal △ of ABC.

Let O be the orthocentre.

Then X, B, Z, O are concyclic, ∵ $O\hat{Z}B$ and $O\hat{X}B$ are each right.

$$\therefore \quad O\hat{X}Z = O\hat{B}Z.$$

$$\text{Similarly} \quad O\hat{X}Y = O\hat{C}Y.$$

But $O\hat{B}Z$ and $O\hat{C}Y$ are each the compt. of $B\hat{A}C$;

$$\therefore \quad O\hat{X}Z = O\hat{X}Y.$$

And similarly for $\wedge^s$ at Y and Z.

Cor. (1). O is the in-centre of △ XYZ.

Cor. (2). The sides of △ ABC are the external bisectors of $\wedge^s$ of △ XYZ.

∴ A, B, C are the ex-centres of △ XYZ.

Cor. (3). △ XYZ is also the pedal △ of the $\triangle^s$ OAB, OBC, OCA—the respective ortho-centres of these $\triangle^s$ being C, A, B.

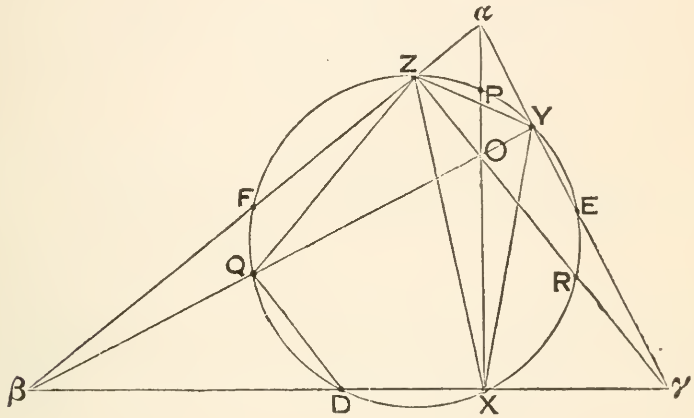
Note—In order to find the circum-centre of the pedal triangle it will be necessary to investigate some of the properties of its circum-circle. This circle is one of the most (perhaps *the* most) curious and fertile of all the circles associated with the original triangle. We shall find that the circum-centre of the pedal triangle is half way between the orthocentre and circum-centre of the original triangle.

THEOREM (20)—(*Poncelet's*) *With respect to any triangle—*

1°, *the circle which circumscribes its pedal triangle, goes through the mid points of the joins of its orthocentre and corners, and the mid points of its sides;*

2°, *the centre of this circle is collinear with its orthocentre, its circum-centre, and its centroid; and bisects the join of the two former;*

3°, *the radius of this circle is half the radius of its circum-circle.*



Let $\alpha\beta\gamma$ be a △; X, Y, Z the feet of its altitudes drawn from α, β, γ respectively; O its orthocentre.

Let ⊙ round XYZ cut αO, βO, γO in P, Q, R respectively: join ZQ.

$$\text{Since } O\hat{Z}\beta \text{ and } O\hat{X}\beta \text{ are each right;}$$

$$\therefore\ O\beta \text{ is diam. of ⊙ through Z and X.}$$

$$\text{Also } Z\hat{Q}O = Z\hat{X}Y = 2\,Z\hat{X}O;$$

$$\therefore\ Q \text{ is centre of ⊙ whose diam. is } O\beta;$$

$$\therefore\ Q \text{ is mid pt. of } O\beta;$$

Similarly R is mid pt. of $O\gamma$; and P is mid pt. of $O\alpha$.

Next let ⊙ round XYZ cut sides again, opposite α, β, γ respectively, in D, E, F: join QD.

$$\text{Then } Q\hat{D}\beta = Q\hat{Y}X, \text{ since QDXY is cyclic,}$$

$$= O\hat{\gamma}X, \text{ since OY}\gamma\text{X cyclic;}$$

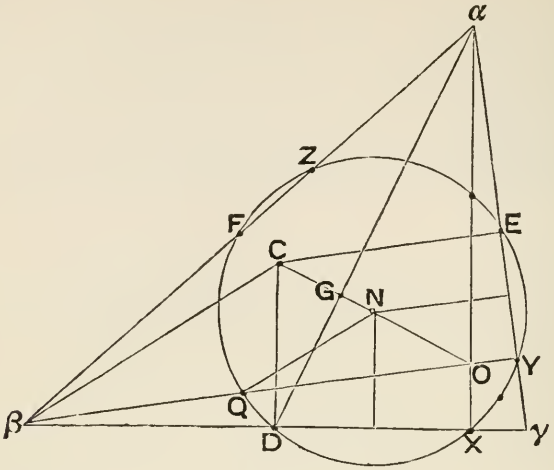
$$\therefore\ QD \text{ is } \parallel \text{ to } O\gamma.$$

$$\text{And since Q is mid pt. of } O\beta,$$

$$\therefore\ D \text{ is mid pt. of } \beta\gamma;$$

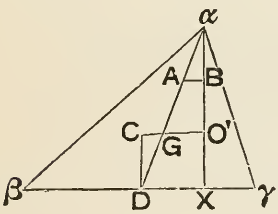
Similarly E is mid pt. of $\gamma\alpha$; and F is mid pt. of $\alpha\beta$:

i. e. 1°, the nine pts. X, Y, Z, P, Q, R, D, E, F are concylic.



Again, the circum-centre C is the intersection of $\perp^s$ at D, E to $\beta\gamma$, $\gamma\alpha$: and the centre of the ⊙ round XYZ is the intersection of the $\perp$ bisectors of its chds. DX, EY.

But each of these last lines bisects CO [i. *Addenda* (15)]; ∴ their pt. of meeting must be N, the mid pt. of CO.



Next let G be the centroid of $\alpha\beta\gamma$, so that

$$G\alpha = 2\,GD.$$

Join CG; and produce it to meet αX in O′

Take A the mid pt. of Gα, and B the mid pt. of O′α; and join AB.

Then AB is ∥ to GO′, and $= \frac{1}{2}$ GO′;

∴ in Δ^s DCG, αBA, we have

$$\left\{\begin{aligned} DG &= A\alpha, \\ C\hat{D}G &= \text{altern. } A\hat{\alpha}B, \\ \text{and } C\hat{G}D &= A\hat{G}O' = \alpha\hat{A}B; \end{aligned}\right.$$

$$\therefore \quad CG = AB = \tfrac{1}{2}\,GO'.$$

Similarly if CG meet βY in O″, we should get

$$CG = \tfrac{1}{2}\,GO''.$$

But C and G are fixed pts;

$\therefore$ O′ and O″ must be the same pt.

i. e must be O, the orthocentre ;

i. e. 2°, O, N, G, C are collinear ;

Lastly : since N is mid pt. of CO,

and Q is mid pt. of βO ;

$\therefore$ NQ $= \frac{1}{2}$ Cβ :

i. e. 3°, rad. ⊙ round XYZ $= \frac{1}{2}$ rad. circum. ⊙.

Cor. CG = 2 GN.

Def. This circle is called the **Nine-point circle.**

Note (1)—We shall use the contraction N. P. for the words **nine-point.**

Note (2)—It is easily seen that—the N. P. ⊙ of $\Delta\, \alpha\beta\gamma$ is also the N. P. ⊙ of each of the Δ^s O$\alpha\beta$, O$\beta\gamma$, O$\gamma\alpha$.

Note (3)—A simple way of treating the N. P. ⊙ is to start with the nine points D, E, F, X, Y, Z, P, Q, R. Then it is easily seen that PFDR, PQDE, are rectangles, having a common diagonal PD ; and thence, immediately, that the nine points are on the ⊙ ; that the intersection of the three diagonals is its centre ; and that this centre is the mid point of CO.

Def. If any and every point on a line, or group of lines (straight or curved) and no other point, satisfies an assigned condition, that line, or group of lines, is called the **Locus** of the point satisfying that condition.*

FROM GEOMETRICAL RESULTS ALREADY GIVEN, THE LOCUS OF A POINT UNDER ANY ONE OF THE FOLLOWING CONDITIONS IS AT ONCE OBVIOUS.

(α) *Condition*—distance from a fixed point constant.

Locus—the circle whose centre is the fixed point, and radius the constant distance.

(β) *Condition*—distance from a fixed straight line constant.

Locus—two straight lines parallel to the fixed line, on opposite sides of it, and at distances from it, each of which is equal to the constant distance.

(γ) *Condition*—distance from a fixed circle constant.

Locus—two circles concentric with fixed circle, and whose radii are the sum and difference of the constant distance and radius of fixed circle ; unless the constant distance is greater than the fixed radius, when the second part of the Locus has no existence.

* Syllabus, p. 19.

(δ) *Condition*—equidistance from two fixed points.

Locus—the straight line bisecting, and perpendicular to, the join of the fixed points.

(ϵ) *Condition*—equidistance from two fixed lines.

Locus—(1) when lines are parallel, the line lying halfway between the fixed lines;

(2) when lines intersect, the two bisectors of the angles between the fixed lines.

(ζ) *Condition*—point to be the vertex of a triangle of constant vertical angle, and on one side of a fixed base.

Locus—the arc of the segment of a circle, whose chord is the fixed base, and whose angle is equal to the constant vertical angle.

(η) *Condition*—sum of squares of its distances from two fixed points constant.

Locus—a circle whose centre is the mid point of the join of the two fixed points.

(θ) *Condition*—difference of squares of its distances from two fixed points constant.

Locus—the straight line perpendicular to the join of the fixed points, through that point in the join, dividing it into parts, the difference of the squares on which is equal to the constant difference.

(ι) *Condition*—that point is the vertex of a triangle on one side of a fixed base, and of constant area.

Locus—the line parallel to the fixed base, at a distance from it such that the rectangle under this distance and the fixed base is double of the constant area.

(κ) *Condition*—the sum of the squares on its distances from the three corners of a fixed triangle constant.

Locus—a circle whose centre is the centroid of the triangle.

(λ) *Condition*—the sum of the squares on its distances from any number of fixed points constant.

Locus—a circle whose centre is the mean centre of the points, for a system of equal multiples.

(μ) *Condition*—with the notation of *Theorem* (20) p. 113, $\Sigma(\mathsf{a} \,.\, \mathsf{AP}^2)$ constant.

Locus—P is on a circle, centre M.

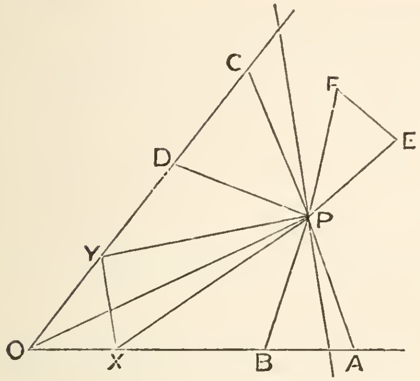
(ν) *Condition*—the feet of perpendiculars from point on the sides of a triangle collinear.

Locus—the circum-circle of the triangle.

(ξ) *Condition*—the orthocentre and circum-circle of a triangle fixed.

Locus—the mid points of the sides, and the corners of its pedal triangle, have a common Locus, viz. the N. P. circle.

To find the Locus of the common vertex of any number of triangles, each of whose bases is fixed in magnitude and position, and the sum of whose areas is constant.



1°, take the case of 2 △s PAB, PCD, on fixed bases AB, CD, and such that △ PAB + △ PCD is constant.

Let AB, CD produced meet in O. In OA take X, and in OC take Y, so that OX = AB, OY = CD.

Join PX, PY, PO, XY.

Then △ PXO + △ PYO = △ PAB + △ PCD :

i. e. △ PXY + △ XOY is const.

But △ XOY is fixed in every respect.

∴ △ PXY is of const. area, and on a fixed base XY.

∴ Locus of P is a ∥ to XY.

When P is on that part of the line *outside* $\widehat{AOC}$, it is necessary to consider one of the △s *subtractive*, and the other *additive*; so that their sum is what has been defined (p. 110) as an *algebraic sum*.

If AB, CD are ∥, the preceding investigation will need modification; and it will be found that the Locus may be *indeterminate* (cf. p. 297) or even *impossible*.

2°, take a third △ PEF, and suppose that

△ PAB + △ PCD + △ PEF is const.

Then, by what precedes, we have

△ PXY + △ PEF const.

But this is the case previously investigated.

And the process may clearly be extended to any number of △s.

Hence the required Locus is a st. line.

Cor. If any number of lines are given in position, the Locus of a point, the sum of whose distances from the lines is constant, reduces at once to the preceding, by taking equal segments on the lines, joining their extremities to the point, and using i. *Addenda* (29).

EXERCISES ON BOOK iii.

NOTE—*Of the following Exercises,* 1–106 *are Theorems to be proved; and depend mainly on the principles of Book* iii: *the remainder are Examples of Loci.*

1. If two circles touch (externally or internally) any line through their point of contact cuts off similar segments.

2. If two equal circles cut, and through one of the points of section any line is drawn, terminated by the circles, the joins of its extremities with the other point of section are equal.

3. If on a side AB, of a triangle ABC, as diameter, a circle is described, and a diameter XY is drawn parallel to BC; then XB, YB bisect the angle ABC internally and externally.

4. If the radius of one circle is the diameter of another, then any line drawn from the point of contact of the circles to meet the outer, is bisected by the inner.

5. If a parallelogram circumscribes a circle its four sides must be equal; and if a parallelogram is inscribed in a circle its four angles must be right.

NOTE—*See Defs. of Book* iv.

6. When two circles intersect, their common chord bisects their common tangent.

7. If any two chords of a circle cut at right angles, the sum of the squares on their segments is equal to the square on the diameter.

NOTE—*Drop* $\perp^s$ *from the centre on the chds., and use* ii. 9.

8. If X, Y are the feet of perpendiculars from A, B on opposite sides of triangle ABC, and BZ is perpendicular to XY (produced if necessary), then angle ABY is equal to angle XBZ.

9. P is a point on the circumference of a circle (centre C), the tangent at P meets a radius CA in T, and PN is perpendicular to CT; then AP bisects the angle TPN.

10. If two circles touch externally, and a line is drawn cutting both, and the four points of section joined to the point of contact; then the angle between the extreme joins is supplementary to the angle between the intermediate joins.

11. If two circles intersect, and through a point of section two lines are drawn cutting the circles again in four points, and the pair of points on each circle are joined; then the joins produced cut at a constant angle.

12. If from the point of contact of two equal circles a chord is drawn in each circle, so that they are at right angles, then the join of their other extremities is equal and parallel to the join of the centres.

13. The bisectors of an angle, and of the opposite external angle of a cyclic quadrilateral, meet on the circumference of its circum-circle.

14. Two equal circles cut in A, B ; if BP, BQ are chords of the circles, each of which is equal to BA ; then PA, QA touch the circles.

15. If a chord of a circle is bisected by another chord, and this again by another, and so on ; the points of bisection continually get nearer the centre.

16. If any number of triangles on the same base, and on the same side of it, have equal vertical angles, the bisectors of these angles are concurrent.

17. If two circles cut, and from any point P on one of them, lines are drawn through the points of section to meet the other in X, Y ; then XY is parallel to the tangent at P.

18. If two circles cut, and through any point in their common chord, or its production, two chords are drawn, one in each circle ; then the extremities of these chords are concyclic.

19. If from a point P on a circle, any chord PA, and the tangent PT are drawn ; and if TXY is any parallel to PA which meets the circle in X, Y ; then triangles PTX, AXP are equiangular to each other.

20. If AB, CD are parallel diameters of two circles, and AC cuts the circles again in P, Q ; then the tangents at P, Q are parallel.

21. If the mid points of adjacent sides of a cyclic quadrilateral are joined, the circles round the four triangles thus formed are equal, and touch the circle round the quadrilateral.

NOTE—*If* $\wedge^s$ *in segts. on equal chds. are equal, the segts. are parts of equal* $\odot^s$.

22. If the diagonals of a quadrilateral are at right angles, then the feet of perpendiculars from the intersection of the diagonals on the sides are concyclic.

23. If AB is a fixed chord of a circle, and PQ any other chord bisected by AB, then the tangents at P, Q meet on a fixed circle.

NOTE—*It will be found that the tangents meet on the* $\odot$ *through* A, B *and the centre of the original* $\odot$.

24. The three circles which go through two corners of a triangle and its orthocentre, are each equal to the circum-circle.

NOTE—*See note on Exercise* 21.

25. If from the extremities of a diameter of a circle perpendiculars are dropped on any chord, their feet are equidistant from the centre.

NOTE—*Drop a* $\perp$ *from the centre on the chd. and use* i, *Addenda* (15).

26. If two circles have external contact at O; and AOB, COD are any lines through O—A, C being on one circumference, and B, D on the other—then AC, BD are parallel.

27. If a cyclic quadrilateral has its diagonals at right angles, the sum of the squares on a pair of opposite sides is equal to the square on the diameter of the circle round it.

NOTE—*See Exercise* 7.

28. If AB, CD are chords of a circle (centre O) cutting at right angles in X—then

$$AB^2 + CD^2 + 4\,OX^2 = 8\,OA^2.$$

NOTE—*Use* ii. *Addenda* (12) *and preceding Exercise.*

29. In a cyclic quadrilateral whose diagonals are at right angles, the distance of the centre of the circum-circle from a side is half the opposite side.

NOTE—*Use Brahmegupta's Theorem*, p. 168.

30. In a cyclic quadrilateral whose diagonals are at right angles, the feet of the perpendiculars from the intersection of the diagonals on the sides, and the mid points of the sides, are eight concyclic points.

31. Two circles cut at A, B; a chord CAD is drawn terminated by the circles; CE, DE are tangents at C, D: then B, C, D, E are concyclic.

32. AB, AC are chords of a circle; X, Y the respective mid points of those arcs they cut off which lie outside angle BAC: then XY cuts off equal parts from AB, AC.

33. Any number of circles touch at a common point, and any line is drawn from that point to cut the circles; in each circle the radius is drawn to the point (not the common point) where it is cut by the line: then these radii are all parallel.

34. AB is any finite line, CD a chord of a circle parallel to AB; AC, being joined, cuts the circle in E; and BE cuts the circle in F: prove that DF cuts AB in a fixed point which is the same for all chords.

35. If two circles intersect, any two parallels through the points of section, terminated by the circles, are equal.

36. X, Y are the feet of the perpendiculars from A, B on opposite sides of triangle ABC; M is the mid point of AB: then angle MXY and angle MYX are each equal to angle C.

37. If the tangents from one point to any number of intersecting circles are equal, all the common chords of the circles are concurrent in that point.

38. In any triangle ABC, if X, Y, Z are the feet of the altitudes from A, B, C respectively; and O is the orthocentre; then

$$AO\,.\,OX = BO\,.\,OY = CO\,.\,OZ.$$

39. From any point P, in a diameter AB of a circle, PQ, PR are drawn, on the same side of AB, so that $\widehat{APQ} = \widehat{BPR}$; then triangles APQ, RPB are equiangular to each other.

40. A variable circle goes through the corner O of a fixed angle, and meets its containing lines in X, Y; if OX + OY is constant, then the circle goes through a fixed point.

NOTE—*See* iii, *Addenda* (15) (γ).

41. Two circles cut in A, B; if from any point P on the circumference of one (whose centre is O) PAX, PBY are drawn to meet the other in X, Y; then XY, PO are at right angles.

42. AB is the diameter of a semi-circle; P, Q are any points on its arc; if AP, BQ meet in X, and AQ, BP in Y; then—

1°, XY is perpendicular to AB;

2°, the tangents at P, Q meet in the mid point of XY.

NOTE—*See* i. *Addenda* (25).

43. If the common tangents of two intersecting circles are met by their common chord produced in X, Y; then

$$XY^2 = (\text{a common tang.})^2 + (\text{common chd.})^2$$

NOTE—*Apply Cor.* ii. 4 (a) *to* XY^2.

44. If two circles are in contact, and there is drawn any pair of parallel diameters, and the ends of these are joined—transversely when contact is external, but directly when internal—then the joins go through the point of contact.

45. Two circles (centres A, B) cut in C; through C are drawn PCX, QCY equally inclined to the line of centres, so that P, Q are on circumference, centre A, and X, Y on circumference centre B; then PX = QY.

NOTE—*From* A, B *drop* $\perp^s$ *on* PX, QY; *and from* B *drop* $\perp^s$ *on* $\perp^s$ *from* A.

46. From any point on the circumference of one of two intersecting circles lines are drawn through both the common points of the circles; if the points in which these lines meet the other circumference are joined, the join is invariable in length.

47. AX, AY are two fixed lines of indefinite length; AB a terminated line bisecting angle XAY; if any circle is drawn to have AB a chord, and X, Y are the points where it cuts the other two lines, then AX + AY is constant.

NOTE—*Drop* BM, BN $\perp^s$ *on* AX, AY; *and prove that* XM = YN.

48. If two circles have internal contact, and a chord of the outer is a tangent to the lesser, then the segments of the chord, made by its point of contact, subtend equal angles at the point of contact of the circles.

49. The perpendicular from the mid point of a side of a triangle on the opposite side of its pedal triangle, bisects that side.

NOTE—*See* iii. *Addenda* (19).

50. If an angle is of fixed magnitude, and each of the lines which form it passes through a fixed point, then its internal bisector will go through one fixed point; and its external bisector through another.

NOTE—*Draw* ⊙ *thro. given pts. and vertex of moving* $\wedge$.

51. The exterior common tangents to two circles, having external contact, also touch the circle on the join of their centres as diameter.

52. If three circles (centres A, B, C) are so placed that the two with centres A, B, have internal contact at P; the two with centres A, C have external contact at Q; and the two with centres B, C, have internal contact at R; then the angle ACB is twice the angle QPR.

NOTE—*Produce* PQ *to meet* BR.

53. If two equal circles are so placed that the tangent to either from the centre of the other is equal to a diameter; then they have a common tangent, which is equal to a radius.

54. If the produced altitudes of a triangle ABC meet its circumcircle in a, b, c; and if any line through the orthocentre meets the sides of the triangle in α, β, γ; then, α being on BC, a on the altitude from A, &c.; aα, bβ, cγ concur on the circumcircle.

55. If a circle circumscribes a triangle, and each of its segments, outside the triangle, is supposed turned about the side which is its chord, as a hinge, until it is again in the plane of the paper; then these three segments will go through one point.

56. ACB is the diameter of a circle (centre C) and PCQ is any sector on a constant arc PQ; if AP, BQ cut in O, then the angle O is constant.

57. If four circles are described, all outside, or all inside, any quadrilateral, so that each of them touches three of its sides (produced when necessary), then their centres are concyclic.

NOTE—*The centre of each circle is the intersection of the bisectors of the angles formed by the three sides it touches.*

58. If from a fixed point, outside a fixed circle, any two lines APQ, ARS are drawn, making equal angles with the diameter through A, and cutting the circle in P, Q and R, S respectively; then if O is the intersection of PS, QR, O is a fixed point.

NOTE—*Take centre* C, *and show that* P, O, C, Q *are concyclic.*

59. AB is a fixed chord of a circle, and O a fixed point in AB; XOY is any

other chord; if C, the mid point of AB, is joined to M, the mid point of XY, then the angle CMX is constant.

60. In a triangle ABC, if X, Y, Z are the feet of the altitudes, O the orthocentre, S the circumcentre, R the circum-radius, and P *any* point in the plane of the triangle, the circles round PXA, PYB, PZC cointersect in a point Q, such that $PO \cdot OQ = \frac{1}{2}(R^2 - SO^2)$.

NOTE—*Use* iii. *Addenda* (13) *and Exercise* 38 *on p.* 182.

61. If two triangles are equiangular to each other; then, of the sides containing two equal angles, the rectangle under a non-corresponding pair (cf. p. 193) is equal to the rectangle under the other pair.

NOTE—*Place the* $\triangle^s$ *so that the equal* $\wedge^s$ *are vertically opposite, and the non-corresponding sides in the same line*; *and use* iii. 35.

62. If on the sides of a triangle as chords circles are described, such that the sum of the angles in their segments remote from the triangle is equal to two right angles, these circles will have one point in common.

63. If from a fixed point P, outside a circle, centre C, tangents PA, PB are drawn, and a third tangent at a variable point T, on the lesser arc AB, meeting PA, PB in Q, R; then for all positions of T—

1°, the perimeter of $\triangle$ PQR is constant: 2°, the angle QCR is constant.

64. If in any triangle ABC, AX is drawn to meet BC in X, so that the angle BAX is equal to the angle C; then $BA^2 = BC \cdot BX$.

65. If in the diameter of a circle, and its production, points P, Q are taken, on opposite sides of the centre C, so that $CP \cdot CQ = (\text{radius})^2$; then any circle through P, Q bisects the circumference of the original circle.

66. Three circles have external contact at P, Q, R; if PQ, PR are produced to meet the circle through Q, R in X, Y, then XY is a diameter of that circle, and is parallel to the line of centres of the other two circles.

67. If circles are described on the sides of a quadrilateral as diameters, the common chord of any adjacent two is parallel to the common chord of the other two.

68. If fixed parallel tangents to a circle are cut by a variable tangent, the part of the variable tangent intercepted between the fixed tangents subtends a right angle at the centre.

69. Two perpendicular radii of a circle, when produced, are cut by a tangent; if tangents are drawn from the points of section they are parallel.

70. If from any point on the circumference of a circle, a perpendicular is drawn to a fixed diameter, and the angle between this perpendicular and the radius to the point bisected; then the bisector will always go through one of two fixed points.

71. The radius of one circle is the diameter of another; if through the centre of the lesser a chord of the larger is drawn perpendicular to the common diameter; and through a point where this chord cuts the lesser, another chord of the larger is drawn perpendicular to the first chord; then the segments of the chords are equal, each to each.

72. If the sides of a quadrilateral touch a circle, so that an opposite pair are parallel; then the line through the centre, parallel to the parallel pair, and terminated by the other pair, is one-fourth the perimeter of the quadrilateral.

NOTE—*See* i. *Addenda* (13), (14), *and* iii. *Addenda* (10).

73. From a fixed point T, outside a circle, tangents TA, TB are drawn; if any point P is taken in the greater arc AB, then the sum of the angles TAP, TBP is constant.

74. If any number of circles touch a fixed line at a fixed point, the tangents at the points where they are cut by a parallel to the fixed line all touch a fixed circle.

NOTE—*Take one* ⊙; *and prove that* $\perp^s$ *from the fixed pt. on the fixed* $\|$, *and on the tangs. at the pts. where it cuts the* ⊙, *are equal.*

75. If a triangle is turned about one corner (considered as vertex) until one of the sides meeting at the vertex is in the same line as the other previously was; then the join of the vertex with the intersection of the two positions of the base produced, bisects the angle between these positions.

NOTE—*If* △ ABC *is turned into position* AB′C′, *so that* BAC′ *is a st. line; and if* BC, C′B′ *meet in* X; *then* ACXC′ *is a cyclic quad.*

76. From each of the three points of contact of the in-circle perpendiculars are dropped on the joins of the other two: if the feet of these perpendiculars are joined, the latter joins are parallel to the sides of the original triangle.

77. If two finite lines AB, CD (when produced) meet in O, and if P is the other point of section of the circles round AOD, BOC; then the triangles PAB, PDC are equiangular.

78. A variable circle goes through two fixed points A, B, and cuts a fixed circle in variable points P, Q: if AP, AQ meet the fixed circle again in p, q respectively, then pq goes through a fixed point.

NOTE—*Let* pq *meet* AB *in* O: *join* PQ. *Then it can be shown that* O *is concyclic with* B, Q, q; *and* ∴ AB . AO *const.*

79. The circumferences which have for chords the sides of a cyclic quadrilateral, intersect again in four concyclic points.

80. If each corner of one quadrilateral is on a side of another, so that each side of the inner is equally inclined to the pair of conterminous sides of the outer which it meets; then the inner is cyclic.

81. On the sides of a triangle segments of circles are described *internally*, and such that the angle in each segment is the supplement of that angle of the triangle opposite the side which is its chord: then the circles—

1°, pass through one point; 2°, are equal; 3°, have their chords of intersection respectively perpendicular to the opposite sides of the triangle.

82. The difference of the squares on the tangents from any point to two concentric circles, is equal to the square on the tangent from any point on the outer circle to the inner.

*83. If a quadrilateral, in a fixed circle, has one diagonal AB fixed in length and position, and the other diagonal fixed in length only; then the intersections of its pairs of opposite sides lie on fixed circles through A, B.

84. The centre C of one circle is on the circumference of another circle, and the circles cut in A, B; from a point P, on the circumference through C, PXB, PAY are drawn to meet the other in X, Y; then AX is parallel to BY.

NOTE—*Join* XY; *and notice that tangs. at* A, B *to* ⊙ (*centre* C) *meet on the other* ⊙.

85. A point P is taken in a line AB, and a point Q in a line AC; if PM, MX are perpendiculars on AC, AB, and QN, NY on AB, AC, then PQ, XY are parallel.

86. A finite line is divided into two parts, and semi-circles are described on the line, and on each of its parts, all on the same side of the line; from the point common to the two inner semi-circles a perpendicular is drawn to the line, and the point where this meets the outer is joined to the extremities of the line, cutting the two inner; then, if the points of section are joined, the join is a common tangent.

87. If all the sides of a cyclic quadrilateral touch a circle, the joins of the opposite points of contact are at right angles.

88. If a line makes equal angles with one pair of opposite sides of a cyclic quadrilateral, it makes equal angles with the other pair.

NOTE—*Use* iii. *Addenda* (6) (7).

89. AB is the diameter of a circle; AP, AQ are any chords meeting the tangent at B in X, Y, respectively: then angle XPY is equal to angle XQY.

NOTE—*Join* PQ, PB.

90. In a cyclic polygon, of an even number of sides, the sum of its alternate angles and of two right angles, is equal to as many right angles as there are sides.

NOTE—*Join a corner of pol. with the 3rd corner from it on each side, and with each alternate corner afterwards; and use* iii. 22.

91. If any point in the diameter of a circle is joined to the extremities of any parallel chord, then the sum of the squares on the joins is equal to the sum of the squares on the segments of the diameter.

92. If two circles cut in A, B; and any points P, Q are taken on one circle, from which PAX, PX′B and QBY, QY′A are drawn, meeting the other circle in X, X′ and Y, Y′ respectively; then XY is parallel to X′ Y′.

NOTE—*Join* XX′, XY′, YY′, YX′; *and use* iii. *Addenda* (5) *and* iii. 21.

93. P is any point within a circle; AB the chord bisected in P; TQ, TR tangents whose chord of contact goes through P; then, if AB cuts the tangents in X, Y, PX is equal to PY.

NOTE—*If* Q, R *are the pts. of contact,* QR *is the chord of contact.*

94. A is a fixed point on the circumference of a circle, BC any chord; if BP, CP make angles with AB, AC respectively equal to those which BC makes with them, then P lies on the diameter through A.

95. If OFAE is a parallelogram, and BOC any line cutting AF, AE in B, C, then

$$BA \,.\, AF + CA \,.\, AE = AO^2 + BO \,.\, OC.$$

NOTE—*Let* AO *meet* ⊙ *round* ABC *in* D: *draw* FX, EY, *meeting* AO *in* X, Y, *so that* $A\hat{F}X = A\hat{D}B$, *and* $A\hat{E}Y = A\hat{D}C$: *then prove* AX = OY; *and use* iii. *Addenda* (5) *and* iii. 36. *Cor.* (*a*).

96. If A, B, C, D and a, b, c, d are two sets of four points, so placed that any two of the first set together with the corresponding two of the second set are concyclic; then if the first set are concyclic, so are the second set.

97. Any four lines are drawn from a point O, so that the two outside angles formed are equal, and are cut by two lines AXBC, aXbc (corresponding letters lying on the same line) then, if angles OAX, OXb are each right, so is OBc.

98. A circle A is inscribed in an isosceles triangle, and a circle B touches its sides at the extremities of the base—then tangents to A, at the points where the circles cut, will meet on circumference of B.

99. If the centre X of one circle is on the circumference of another circle Y; and from any point A, on circumference of Y, a tangent is drawn to X, meeting Y in B; from B another tangent to X, meeting Y in C; and from C another tangent to X, meeting Y in D; then DA is also a tangent to X.

100. The circles circumscribing the four triangles, formed by four intersecting lines, go through one point; which is concyclic with their centres.

NOTE—*Let* XAB, XDC, YCB, YDA *be the lines, forming quad.* ABCD; *let* ⊙s *round* BCX, ABY *cut in* O: *then* O *can be shown concyclic with* A, D, X *and with* C, D, Y.

101. From A, one of the points of intersection of two circles, two lines AXY, APQ are drawn at right angles; if X, Q are on one circle, and Y, P on the other, then

$$PQ^2 + XY^2 = 4 \text{ (sq. on join of their centres).}$$

NOTE—*Use* ii. *Addenda* (12).

102. If the circle through B, C and the orthocentre of a triangle ABC, meets the median from A (produced) in X, then AX is twice that median.

103. If the sides and angles of a triangle ABC, are given, and its position varies, subject only to the condition that AB, AC each go through a fixed point; then BC always touches a fixed circle.

NOTE—*Let* P *be the fixed pt. in* AB, *and* Q *in* AC; *on* PQ *describe a segt. contg. an* $\hat{A}$; *and draw* PO *to cut off segt. contg. an* $\hat{B}$: O *is cent. of the required* ⊙.

104. If four circles are drawn, each passing through three out of four fixed points; then one of the angles between the tangents at the intersection of one pair of circles is equal to one of the angles between the tangents at the intersection of the other pair.

105. If two *Simson's Lines* are drawn with respect to points at the ends of a diameter, then—

1°, the lines are at right angles;

2°, they intersect on the N. P. circle. (*Tucker*: *E. T. Reprint*; III.)

NOTE—*If* PCQ *is diam.*, PM, PL, QX, QY $\perp^s$ *on sides* $\alpha\beta$, $\alpha\gamma$; *then* 1°, *comes from considering* $\wedge^s$ *of cyclic quads.* PMαL, QXαY; *and* 2°, *comes from the facts that* N *is mid pt.* CO, *and that* PO, QO *are bisected by the two Simson's Lines*: *cf.* iii. *Addenda* (18) and (20).

106. If ABC is a triangle, and P any point, the N. P. circles of the triangles PAB, PBC, PCA, intersect in a point X, on the N. P. circle of ABC.

If P is on the circum-circle of ABC, then—

1°, the *Simson's Lines* of the triangles PAB, PBC, PCA, ABC, relatively to C, A, B, P, respectively, are concurrent in X; and

2°, the centroids of the same triangles are concyclic. (Prof. Bordage: *Educational Times: Reprint*; Vol. XLVI.)

NOTE—*The first part of the Theorem may be put thus*—

The four triangles determined by any four points, taken three and three together, are such that their N. P. circles have a common point.

Take the N. P. ⊙^{s} *as circumscribing the* Δ^s *formed by the joins of mid pts.*; *and use the converse of* iii. 22.

EXAMPLES OF LOCI.

NOTE—*In each of the following the Locus is to be determined as completely as possible: if the Locus is a straight line, its position with regard to given points or lines—and if a circle, its centre and radius—are to be found.*

107. Four rods are pivoted together, so that the pivots are the corners of a rhombus, and the framework is capable of motion in one plane; if one rod is fixed, and the others are moved about, then the locus of the intersection of the diagonal lines joining the pivots is a circle.

108. The ends of a rod of given length slide on fixed straight intersecting wires; if perpendiculars to the wires at, or from, the ends of the rod are drawn, the loci of their intersection are two circles concentric at the intersection of the wires.

109. The locus of the mid point of all lines drawn from a fixed point to meet a fixed circle, is a circle.

NOTE—*This is a particular case of a useful Theorem given hereafter:* vi. *Addenda* (11).

110. If in a fixed circle a triangle is inscribed, so as to have its orthocentre at a fixed point; then the locus of the mid points of its sides is a circle.

NOTE—*Produce join of orthocent. and one mid. pt. a dist. equal to itself; show that the extremity of this lies on the* ⊙; *and then use the preceding Exercise.*

111. If the extremities of a diameter of a circle are joined transversely to the extremities of a chord of constant length, the locus of the intersection of the joins is a circle.

112. If a chord of a given circle subtends a right angle at a given point (within or without the circle) the locus of its mid point is a circle, whose centre is the mid point of the join of the centre of the given circle to the fixed point.

NOTE—*Use Locus* (η) p. 178.

113. If a quadrilateral circumscribes a circle, the join of the mid points of its diagonals goes through the centre of the circle.

NOTE—*This is a particular case of Newton's Principia, Lib.* I, *Lemma* 25, *Cor.* (3); *and may be easily done by using Locus* on p. 179.

114. Given the base and vertical angle of a triangle, find the Locus of its—
1°, in-centre: 2°, ortho-centre: 3°, centroid: 4°, ex-centres: 5°, N.P. centre.

NOTE—5° *can be made to depend on Ex.* 109 *above.*

115. Two variable circles, each touching the same given line at a given point (different for each circle) also touch each other: find the Locus of the point of contact of the circles.

116. Two opposite corners of a given square move on two lines at right angles: find the Loci of the other corners.

117. Given the base of a triangle in magnitude and position; and given also, 1°, the *sum* of the other two sides, or 2°, their *difference*; find the Loci of the feet of perpendiculars from the ends of the base on the bisector of the external vertical angle for the *sum*, and vertical angle for the difference.

118. Given an angle of a triangle, in position and magnitude, and given also the sum of the sides which contain it; find the Locus of its circum-centre.

NOTE—*Use* iii. *Addenda* (15).

119. Given a circle, and a fixed point within it; find the Locus of the intersection of tangents at the extremities of all chords through the point.

120. Given two fixed circles, 1°, intersecting, 2°, not intersecting; find, in each case, the Locus of the points from which tangents to the circles are equal.

121. Given two fixed points, find the Locus of a variable point whose distance from one of the points is twice its distance from the other.

NOTE—*This is a particular case of an important Locus, given on* p. 294.

122. AB is a fixed chord of a fixed circle, AP a variable chord of the same circle; find the Locus of the mid point of BP.

123. Through a point P within a rectangle ABCD, parallels are drawn to the sides; and P moves so that the difference of the rectangles PA, PC is constant; find the Locus of P.

124. The vertex of an isosceles triangle (given in all but position) moves along the circumference of a fixed circle (whose radius is equal to one of the equal sides) and one of the extremities of the base moves along a fixed diameter of the circle; find the Locus of the other extremity

125. A line of given length moves with its extremities on two fixed intersecting lines; find the Locus of the orthocentre of the triangle thus formed.

126. A point, inside or outside a circle, is joined to two fixed points on its circumference; if the joins intercept a constant arc, find the Locus of the point.

127. AB is a fixed line, O a fixed point, and MN a fixed length; if a variable point P moves so that (PQ being the perpendicular from it on AB)

$$PO^2 = PQ \cdot MN,$$

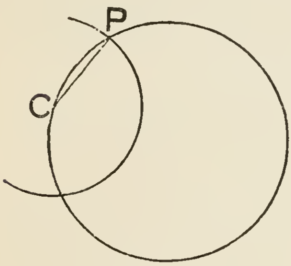
find the Locus of P.

NOTE—*Draw* OX $\perp$ *to* AB, *and produce* XO *to* C, *so that* OC $= \frac{1}{2}$ MN: *then it can be shown that* CP *is constant, and* $\therefore$ *that* C *is the centre of the* $\odot$ *which is Locus of* P. *Cf.* p. 364.

BOOK iv.

Proposition 1.

PROBLEM—*In a given circle to draw (when possible) a chord equal to a given straight line.*



Since no chd. of a ⊙ can be greater than its diam., the prob. will not be possible unless the given st. line is not greater than the diam.

Take any pt. C on the circumf. of given ⊙.

With centre C, and given line as radius, describe a ⊙.

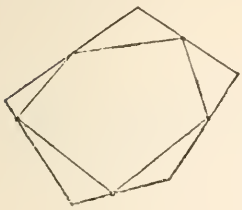
Let P be one of the pts. in which the ⊙s cut.

Join CP.

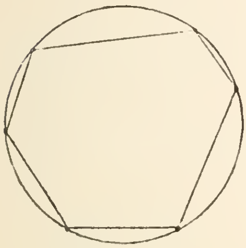
Then CP = given line, and is a chd. of given ⊙.

NOTE—A line drawn (as above) from a given point, to meet a given circle (or line) and be of a given length, is said to be *inflected from the point to the circle (or line).*

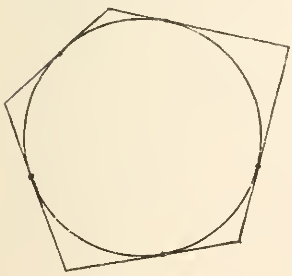
Def. When all the corners of one rectilineal figure are on the sides of another rectilineal figure, the first figure is said to be **inscribed in** the second; and the second figure is said to be **circumscribed about** the first.



Def. When all the corners of a rectilineal figure are on the circumference of a circle, the rectilineal figure is said to be **inscribed in** the circle; and the circle is said to be **circumscribed about** the rectilineal figure.



Def. When each side of a rectilineal figure touches a circle, the rectilineal figure is said to be **circumscribed about** the circle; and the circle is said to be **inscribed in** the rectilineal figure.

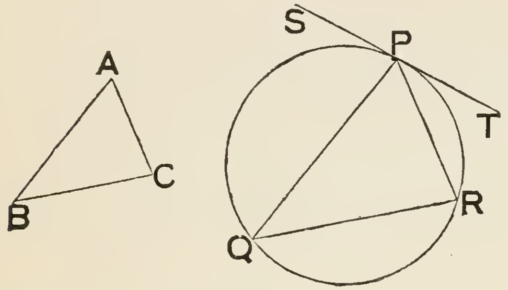


Def. When a rectilineal figure is equiangular and equilateral it is called **regular.**

Def. If the angles of a rectilineal figure (taken successively) are equal respectively to those of another (also taken successively) then the figures are said to be **equiangular to each other**; each angle of the one is said to **correspond** to the angle equal to it in the other; and the sides joining the vertices of corresponding angles are called **corresponding sides.**

Proposition 2.

PROBLEM—*In a given circle to inscribe a triangle equiangular to a given triangle.*



Let ABC be the given $\triangle$.

At any pt. P, on the circumf. of given $\odot$, draw a tang. TPS.

Draw chds. PQ, PR of this $\odot$, so that

$$S\hat{P}Q = \hat{C} \quad \text{and} \quad T\hat{P}R = \hat{B}.$$

Join QR.

Then $T\hat{P}R = P\hat{Q}R$, in altern. segt.;

$\therefore \quad P\hat{Q}R = \hat{B}$.

Similarly $P\hat{R}Q = \hat{C}$;

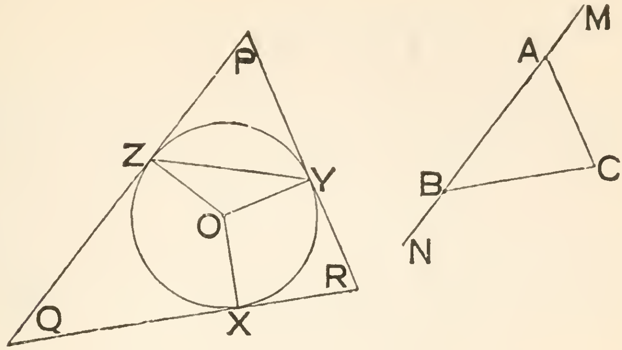
$\therefore$ also remg. $Q\hat{P}R =$ remg. $\hat{A}$:

i. e. $\triangle$ PQR is equiang. to $\triangle$ ABC;

and it is inscribed in given $\odot$.

Proposition 3.

PROBLEM—*About a given circle to circumscribe a triangle equiangular to a given triangle.*



Let ABC be the given $\triangle$; O the centre of the given $\odot$.

Produce BA to M, and AB to N.

Draw any radius OZ of given $\odot$; and radii OX, OY,

so that $Z\hat{O}Y = C\hat{A}M$ and $Z\hat{O}X = C\hat{B}N$.

Draw tangs. to the $\odot$ at Y, Z ; and join YZ.

Then the sum of $\wedge^s$ made by YZ, on side remote from O, with tangs. at Y, Z, $<$ two rt. $\wedge^s$.

$\therefore$ these tangs. will meet on that side—say in P.

Similarly let tangs. at Z, X, meet in Q;

and let tangs. at X, Y, meet in R.

Then, since all $\wedge^s$ of a quad. make up four rt. $\wedge^s$,

and $O\hat{Y}P$ and $O\hat{Z}P$ are each right;

$$\therefore \quad \hat{P} + Y\hat{O}Z = \text{two rt. } \wedge^s,$$

$$= B\hat{A}C + C\hat{A}M.$$

$$\text{But} \quad Y\hat{O}Z = C\hat{A}M;$$

$$\therefore \quad \hat{P} = B\hat{A}C.$$

$$\text{Similarly} \quad \hat{Q} = A\hat{B}C;$$

$$\text{and} \quad \hat{R} = A\hat{C}B:$$

i.e. $\triangle$ PQR is equiang. to $\triangle$ ABC ;

and it is circumscribed about given $\odot$.

Proposition 4.

PROBLEM—*To inscribe a circle in a given triangle.*

Prove (see p. 73) the 1st part of i. *Addenda* (24) viz. that

$$IU = IV = IW.$$

Then ⊙ with I as centre, and any one of these as radius, goes through U, V, W; and touches sides of △ at U, V, W,

∵ ∧s at those pts. are each right:

i.e. that ⊙ is inscribed in the △ $\alpha\beta\gamma$.

Proposition 5.

PROBLEM—*To circumscribe a circle about a given triangle.*

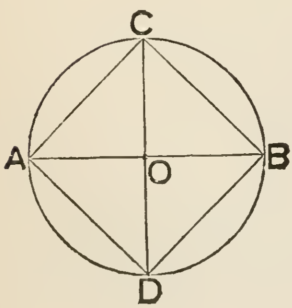
Prove (see p. 72) the 1st part of i. *Addenda* (23) viz. that

$$C\alpha = C\beta = C\gamma.$$

Then ⊙ with C as centre, and any one of these as radius, circumscribes the △ $\alpha\beta\gamma$.

Proposition 6.

PROBLEM—*To inscribe a square in a given circle.*



Let O be centre of given ⊙.

Take any diam. AOB; and draw the diam. COD ⊥ to it.

Join AC, CB, BD, DA.

Then $\wedge^s$ at O, being right, are equal.

$\therefore$ the arcs they subtend are equal.

$\therefore$ also the chds. AC, CB, BD, DA, of these arcs are equal.

$\therefore$ fig. ACBD is equilat.

And its $\wedge^s$ are right,

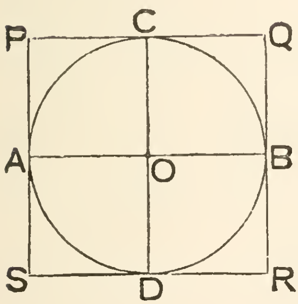
$\because$ each of them is an $\wedge$ in a semi $\odot$.

$\therefore$ fig. ACBD is a square;

and it is inscribed in given $\odot$.

Proposition 7.

PROBLEM—*To circumscribe a square about a given circle.*



Let O be centre of the given $\odot$.

Take any diam. AOB; and draw the diam. COD $\perp$ to it.

Draw tangs. at A, C, B, D; and let P be intersec. of tangs. at A and C, Q of tangs. at C and B, R of tangs. at B and D, and S of tangs. at D and A; so that PQRS is a circumscribing quadrilateral.

Then the lines PCQ, AOB, SDR are $\perp$ to COD.

And the lines PAS, COD, QBR are $\perp$ to AOB.

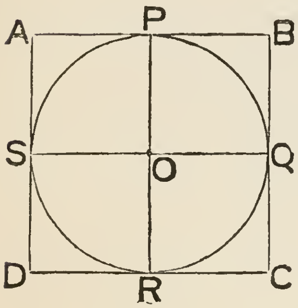
$\therefore$ all the quads. are rects.

And each of the sides of PQRS = a diam. of the $\odot$.

$\therefore$ PQRS is a square.

Proposition 8.

PROBLEM—*To inscribe a circle in a given square.*



Let ABCD be given sq.

Bisect AB in P, and BC in Q.

Draw PR, QS ∥ to sides of sq., and meeting opposite sides in R, S; and let O be their pt. of intersect.

Then all the quads. are rects.

And OP, OQ, OR, OS are each opposite a half side of sq.

∴ OP = OQ = OR = OS.

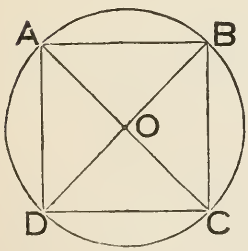
∴ a ⊙ with O as centre, and any one of these lines as radius, will go through P, Q, R, S.

And this ⊙ will be inscribed in given sq.

∵ ∧s at P, Q, R, S are right.

Proposition 9.

PROBLEM—*To circumscribe a circle about a given square.*



Let ABCD be given sq.

Draw its diags. AC, BD, intersecting in O.

Then, since the diags. of a ▭ bisect each other, and the diags. of a sq. are equal,

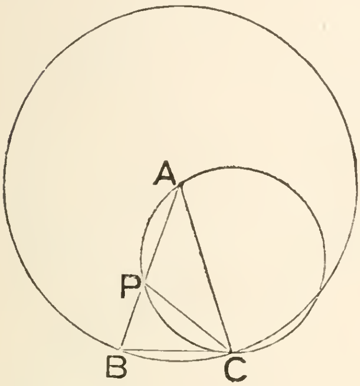
∴ OA = OB = OC = OD.

∴ ⊙ with O as centre, and any one of these as radius, will go through A, B, C, D:

i. e. that ⊙ circumscribes given sq.

Proposition 10.

PROBLEM—*To describe an isosceles triangle having each angle at the base double of the vertical angle.*



Take any st. line AB, and divide it in P, so that

rect. under AB, BP = sq. on AP.

With A as centre, and AB as radius describe a ⊙; and in it place chd. BC equal to AP.

Join AC, PC; and about △ APC describe a ⊙.

Then ∵ sq. on BC = sq. on AP,

= rect. under AB, BP;

∴ BC touches the ⊙ which APB cuts.

And ∵ BC is a tang. and CP a chd. of same ⊙;

∴ $B\hat{C}P = C\hat{A}P$, in altern. segt.

∴ $B\hat{P}C$, which $= C\hat{A}P + P\hat{C}A$,

also $= B\hat{C}P + P\hat{C}A$;

i. e. $= A\hat{C}B$;

∴ also $= A\hat{B}C$, ∵ AC = AB.

∴ CP = CB = PA.

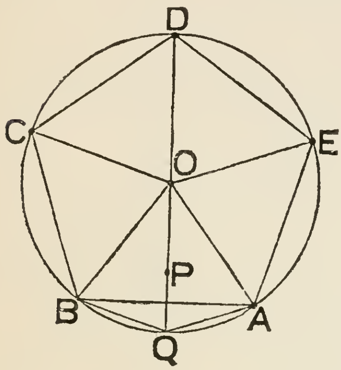
∴ $P\hat{A}C = P\hat{C}A$.

∴ $B\hat{P}C = 2\,P\hat{A}C$:

i. e. △ ABC has $A\hat{B}C$ and $A\hat{C}B$ each double of $B\hat{A}C$.

Proposition 11.

PROBLEM—*To inscribe a regular pentagon in a given circle.*



Let O be the centre of the given ⊙.

Take any radius OQ; and divide it in P, so that

rect. under OQ, QP = sq. on OP.

Draw chds. QA, QB, of ⊙, so that each of them = OP.

Join OA, OB.

Then the construction has given two $\triangle$s OQA, OQB, such that by iv. 10 each of them has its $\wedge$ at O half of each of its remg. $\wedge$s.

Now, since the three $\wedge$s of a $\triangle$ together make up two rt. $\wedge$s,

$\therefore$ $A\hat{O}Q$ and $B\hat{O}Q$ each = one-fifth of two rt. $\wedge$s;

$\therefore$ $A\hat{O}B$ = one-fifth of four rt. $\wedge$s;

i.e. = one-fifth of all consecutive $\wedge$s made by any number of lines meeting in O;

$\therefore$ if we draw successively the radii OC, OD, OE,

so that $B\hat{O}C = C\hat{O}D = D\hat{O}E = A\hat{O}B$,

then fifth remg. $E\hat{O}A$ = either of them.

$\therefore$ five arcs AB, BC, CD, DE, EA are equal.

$\therefore$ chds. of these arcs are equal.

$\therefore$ pentagon ABCDE is equilat.

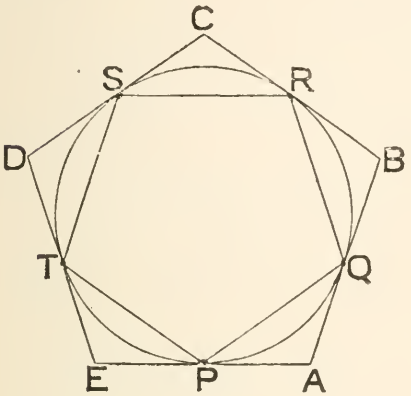
Also it is equiang.,

for each of its $\wedge$s stands on an arc which is treble of arc AB:

i.e. it is regular.

Proposition 12.

PROBLEM—*To circumscribe a regular pentagon about a given circle.*



Let PQRST be a regular pentagon inscribed in given ⊙.

Let tangents to ⊙ at P and Q, at Q and R, at R and S, at S and T, at T and P, meet respectively in A, B, C, D, E, forming a circumscribing pentagon ABCDE.

Since AP, AQ are tangs. and PQ a chd. of ⊙;

$\therefore$ $A\hat{P}Q$ and $A\hat{Q}P$ each $=$ $\wedge$ in altern. segt. cut off by PQ.

Similarly $B\hat{Q}R$ and $B\hat{R}Q$ each $=$ $\wedge$ in altern. segt. cut off by QR.

But since $PQ = QR$;

$\therefore$ $\wedge$ in segt. cut off by PQ $=$ $\wedge$ in segt. cut off by QR.

$\therefore$ $A\hat{P}Q = A\hat{Q}P = B\hat{Q}R = B\hat{R}Q$.

And as also $PQ = QR$;

$\therefore$ $\triangle APQ \equiv \triangle BQR$.

And similarly each of them $\equiv \triangle CRS \equiv \triangle DST \equiv \triangle ETP$.

$\therefore$ $\hat{A} = \hat{B} = \hat{C} = \hat{D} = \hat{E}$.

$\therefore$ pentagon ABCDE is equiang.

And it is also equilat.;

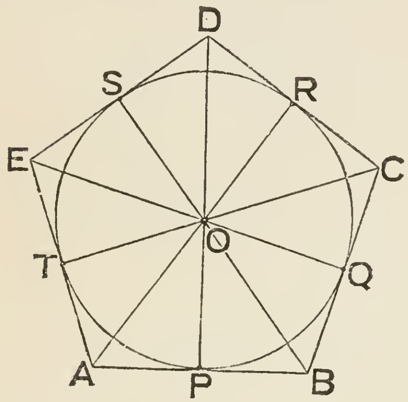
$\because$ $PA = AQ = QB = BR =$ &c. $= EP$,

and $\therefore$ $AB = BC = CD = DE = EA$:

i. e. it is regular.

Proposition 13.

PROBLEM—*To inscribe a circle in a given regular pentagon.*



Let ABCDE be the given regular pentagon.

Let bisectors of $E\hat{A}B$ and $A\hat{B}C$ meet in O.

Join OE, OD, OC.

Then in $\triangle$s EAO, BAO, we have

$$\left.\begin{aligned} AE &= AB, \\ AO &\text{ common}, \\ \text{and } E\hat{A}O &= B\hat{A}O; \end{aligned}\right\}$$

$$\begin{aligned} \therefore\ A\hat{E}O &= A\hat{B}O, \\ &= \text{half } A\hat{B}C, \\ &= \text{half } A\hat{E}D, \text{ since pent. is regular.} \end{aligned}$$

$\therefore$ EO bisects $A\hat{E}D$.

Similarly it could be shown that OD bisects $E\hat{D}C$,

and that OC bisects $D\hat{C}B$.

Now draw OP, OQ, OR, OS, OT,

$\perp$ to AB, BC, CD, DE, EA respectively.

Then in $\triangle^s$ OAT, OAP, we have

$$\left.\begin{array}{r} O\hat{A}T = O\hat{A}P, \\ O\hat{T}A = O\hat{P}A, \\ \text{and } AO \text{ common}; \end{array}\right\}$$

$\therefore \quad OT = OP.$

And similarly each of them $= OQ = OR = OS$.

$\therefore$ ⊙ described with centre O, and radius OP, will go through P, Q, R, S, T;

and will *touch* sides of pentagon at these pts.;

$\because$ its radii make rt. $\wedge^s$ with sides at those pts.:

i.e. this ⊙ will be inscribed in pentagon.

EXERCISES ON iv. 10.

If CP meets large ⊙ in Q, and CA meets it in R; and if S is the second point where ⊙s cut, and BC, AS meet in T; then, making the necessary joins—

(1) $\triangle$ ACS $\equiv \triangle$ ABC $\equiv \triangle$ QPA:

(2) $\triangle^s$ SQR, TAB are each equiangular to $\triangle$ ABC:

(3) CPST is a ▱:

(4) If I is in-centre of $\triangle$ ABC, BI = BP.

(5) Circum-centre of $\triangle$ BPC is on arc CP.

(6) (Radius of circum-⊙ of $\triangle$ CPB)^2
= (diameter of smaller ⊙)^2 − (radius of larger ⊙)^2.

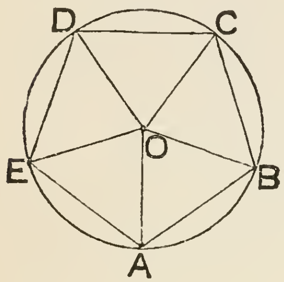
(7) CP is a side of regular pentagon in smaller ⊙:

(8) BC, BQ, AB are respectively equal to sides of regular decagon, pentagon, and hexagon (see Prop. 15) in larger ⊙.

(9) If CX is $\perp$ on AB, $AX^2 + BX^2 = 2\,CX^2$.

Proposition 14.

PROBLEM—*To circumscribe a circle about a regular pentagon.*



Let ABCDE be a regular pentagon.

Let bisectors of $\hat{EAB}$ and $\hat{ABC}$ meet in O.

Join OC, OD, OE.

Then in $\triangle^s$ EAO, BAO, we have

$$\left.\begin{array}{r} AE = AB, \\ AO \text{ common}, \\ \text{and } E\hat{A}O = B\hat{A}O; \end{array}\right\}$$

$$\therefore \quad A\hat{E}O = A\hat{B}O,$$

$$= \text{half } A\hat{B}C,$$

$$= \text{half } A\hat{E}D, \because \text{ pent. is regular.}$$

$$\therefore \quad OE \text{ bisects } A\hat{E}D.$$

Similarly OD bisects $E\hat{D}C$, and OC bisects $D\hat{C}B$.

Again, since $O\hat{A}B = \text{half } E\hat{A}B = \text{half } A\hat{B}C$;

$$\therefore \quad O\hat{A}B = O\hat{B}A.$$

$$\therefore \quad OA = OB.$$

Similarly each of them = OC = OD = OE.

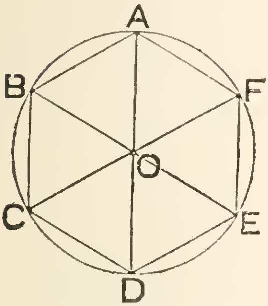
∴ ⊙ with O as centre and any one of them as radius will go through A, B, C, D, E:

i.e. this ⊙ will circumscribe the pentagon.

Note—The construction and proof of the two preceding propositions will apply equally well, if for a regular pentagon there be substituted any other regular polygon: see Theorem (1) of *Addenda* to this Book.

Proposition 15.

PROBLEM—*To inscribe a regular hexagon in a given circle.*



Let O be centre of given ⊙.

Take any radius OA; and draw the chd. AB, equal to OA.

Join OB.

Then △ OAB is equilat.

$\therefore$ $A\hat{O}B$ = one-third of two rt. ∠s,

= one-sixth of four rt. ∠s,

= one-sixth of all the ∠s which can be placed round O successively.

$\therefore$ if radii OC, OD, OE, OF are drawn, so that

$$A\hat{O}B = B\hat{O}C = C\hat{O}D = D\hat{O}E = E\hat{O}F,$$

each of them = remg. $A\hat{O}F$,

and pts. A, B, C, D, E, F divide whole circumf. into six equal parts.

$\therefore$ hexagon ABCDEF is equilat.

Also it is equiang.,

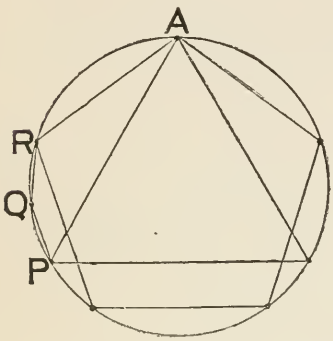
for each of its ∠s stands on an arc which is four times arc AB.

$\therefore$ it is regular;

and it is inscribed in given ⊙.

Proposition 16.

PROBLEM—*To inscribe a regular quindecagon in a given circle.*



Let AP be a side of an equilat. △, inscribed in the ⊙; and AR a side of a regular inscribed pentagon.

Then of such fifteen equal parts as the circumf. can be divided into, beginning at A,

AP, being one-third of the whole, must contain five, and AR, being one-fifth of the whole, must contain three.

∴ PR must contain two such parts.

Bisect PR in Q.

∴ arcs PQ, QR are each one-fifteenth of whole circumf.

∴ if chds. PQ, QR be drawn, and chds. equal to them placed round the ⊙, an equilat. quindecagon will be inscribed in the ⊙.

Also this quindec. will be equiang. for each of its ∧s will stand on an arc which is thirteen-fifteenths of the circumf.

∴ it will be regular.

Note—A regular hexagon, or quindecagon, can be circumscribed about a circle in the same manner as (in Prop. 12) a regular pentagon was circumscribed—viz. by drawing tangents at the corners of the corresponding inscribed figure: see Theorem (2) of *Addenda* to this Book.

ADDENDA TO BOOK iv.

THE FOLLOWING ARE THE MOST OBVIOUS COROLLARIES TO THE PROPS. IN BOOK iv.

iv. 4. (α) Four circles can be drawn to touch each of three lines, unless the lines are concurrent, or parallel, when no circle touches them all. For the remaining three, see i. *Addenda* (27).

(β) If two lines are parallel and a third cuts them, then two circles will touch the three lines.

(γ) An infinite number of circles can be drawn to touch two lines. The *Locus* of the centres of these circles when the lines—

1°, are parallel, is the line lying half way between them ;

2°, intersect, is the pair of bisectors of the angles between them.

iv. 10. The vertical angle of the triangle described by this Prop. is two-fifths of a right angle ; and by means of it a right angle can be divided into five equal parts.

iv. 11. (α) Each diagonal of a regular pentagon is parallel to the side which is not conterminous with it.

(β) All the diagonals of a regular pentagon form, by their intersection, a regular pentagon.

(γ) Every equilateral cyclic polygon is equiangular ; but *not always* conversely: cf. p. 210.

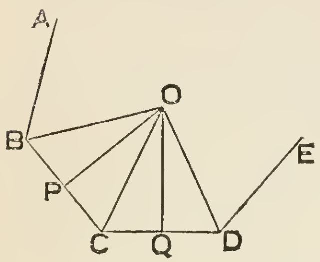
iv. 15. Each side of a regular cyclic hexagon is equal to the radius of its circumscribing circle ; and its area is six times that of an equilateral triangle on that radius.

iv. 6, 11, 15, 16. By bisecting the arcs between each adjacent pair of corners of the polygons inscribed by these Props. and joining each point of bisection to the corners adjacent to it, regular polygons of double the number of sides, in each case, are inscribed in the circles. The same process may be extended : so that this Book gives the modes of inscribing in, or circumscribing about a circle, regular polygons of

$$\left.\begin{array}{llll} 3, & 6, & 12, & 24, \text{ \&c.,} \\ 4, & 8, & 16, & 32, \text{ \&c.,} \\ 5, & 10, & 20, & 40, \text{ \&c.,} \\ 15, & 30, & 60, & 120, \text{ \&c.} \end{array}\right\} \text{sides.}$$

Some Immediate Developments of Book iv.—not so obvious as to be properly called Corollaries.

Theorem (1)—*Any regular polygon may have—*
(α) *a circle described about it;*
(β) *a concentric circle inscribed in it.*



Let AB, BC, CD, DE be 4 consecutive sides of a regr. pol.

Bisect $B\hat{C}D$ and $C\hat{D}E$ by CO, DO; and join BO.

Then in $\triangle^s$ BCO, DCO, we have

$$\left.\begin{aligned} BC &= CD, \\ CO &\text{ common}, \\ \text{and}\quad B\hat{C}O &= D\hat{C}O; \end{aligned}\right\}$$

$$\therefore \quad \triangle BCO \equiv \triangle DCO.$$

$$\therefore \quad OB = OD.$$

But since $O\hat{C}D = O\hat{D}C$, each being half an $\wedge$ of pol.

$$\therefore \quad OC = OD = OB.$$

Also $O\hat{B}C = O\hat{D}C = \frac{1}{2}$ an $\wedge$ of pol.

$\therefore$, as above, all lines from O to corners of pol. are equal.

$\therefore$ (α) a $\odot$ with centre O, and any one of these lines as radius, will circumscribe the pol.

Next, draw OP, OQ respecty. $\perp$ to CB, CD.

Then in $\triangle^s$ CPO, CQO, we have

$$\left.\begin{aligned} P\hat{C}O &= Q\hat{C}O, \\ C\hat{P}O &= C\hat{Q}O, \\ \text{and}\quad CO &\text{ common}; \end{aligned}\right\}$$

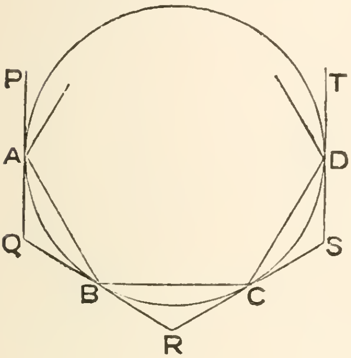
$$\therefore \quad OP = OQ.$$

Similarly all $\perp^s$ from O on sides of pol. are equal.

$\therefore$ (β) a $\odot$, with O as centre and any one of these $\perp^s$ as radius, will go through the feet of all the $\perp^s$; and will touch sides of pol. at those feet, $\because$ the $\wedge^s$ there are right; so that it will be inscribed in pol.

Note—Evidently the area of a regr. pol. is half the rect. under its in-radius (sometimes called its *apothem*) and perimeter.

THEOREM (2)—*If a regular polygon is inscribed in a circle, and tangents to the circle are drawn at its corners, they will form a regular polygon, of the same number of sides, circumscribed about the circle.*



Let A, B, C, D, be 4 consecutive corners of a regr. pol. inscribed in a $\odot$.

At these pts. draw tangs. PAQ, QBR, RCS, SDT.

Then since BC is a secant, and RB, RC tangs. at B, C,

$\therefore$ $\widehat{RBC}$ and $\widehat{RCB}$ each $=$ $\wedge$ in altern. segt. cut off by BC.

Similarly $\widehat{SCD}$ and $\widehat{SDC}$ each $=$ $\wedge$ in altern. segt. cut off by CD.

But since $CB = CD$,

$\therefore$ $\wedge^s$ in these segts. cut off by them are equal.

$$\therefore \quad \widehat{RBC} = \widehat{RCB} = \widehat{SCD} = \widehat{SDC}.$$

$$\therefore \quad \triangle RBC \equiv \triangle SCD.$$

$$\therefore \quad DS = SC = CR = RB;$$

$$\text{and } \hat{R} = \hat{S}.$$

Similarly all the tangs. from pts. P, Q, R, S, T, are equal.

And all $\wedge^s$ at same pts. are equal.

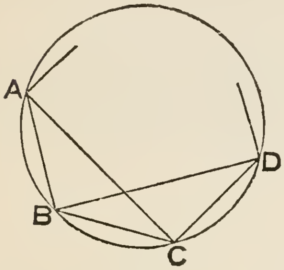
$\therefore$ outer pol. is equiang.

And also equilat. since each of its sides is double one of the equal tangs.

Also all its sides touch $\odot$, by construction.

$\therefore$ it is a regr. circumscribed pol.

THEOREM (3)—*Every equiangular cyclic polygon has its alternate sides equal; and if the number of its sides is odd the sides are all equal; but if even, not necessarily so.*



Let A, B, C, D, &c., be consecutive corners of an equiang. cyclic pol.

Join AC, BD.

Then since $\hat{ABC} = \hat{BCD}$,

$\therefore$ arc ABC = arc BCD.

$\therefore$ arc AB = arc CD.

$\therefore$ AB = CD.

Similarly CD = next altern. side.

And so on round the pol.

Again, it could be shown similarly that

BC = next altern. side,

= next beyond again.

And so on round the pol.

If the pol. have an odd number of sides,

the series of altern. equals beginning with AB,

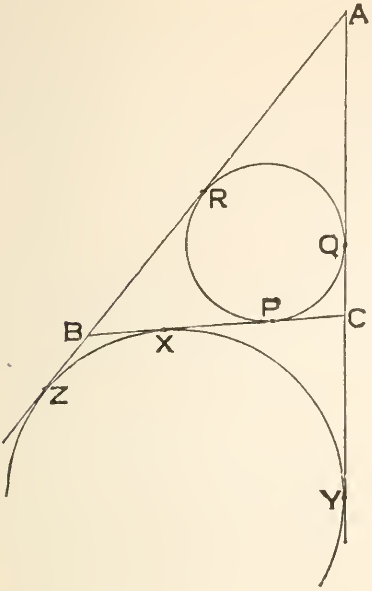
and the series of altern. equals beginning with BC,

will have one side in common;

and then pol. is equilat.

But, if pol. have an even number of sides, the two sets of series will not necessarily be equal.

THEOREM (4)—*If the sides respectively opposite to the corners* A, B, C *of a triangle, are denoted by* a, b, c, *and if* s *denote the semiperimeter of the triangle; then each of the tangents from* A *to the circle exscribed to* BC, *is equal to* s; *each of the tangents from* B *to the same circle is equal to* s – c; *each of the tangents from* A *to the inscribed circle is equal to* s – a; *and similar expressions hold for the other similar tangents.*



Let sides a, b, c respectively touch the in-⊙ at P, Q, R; and the ex-⊙ at X, Y, Z.

Then

$$AY + AZ = AC + CX + AB + BX,$$
$$= AC + AB + BC.$$
$$\therefore \quad AY = s \quad = AZ,$$
$$BZ = s - c = BX,$$
$$CY = s - b = CX.$$

Again $AR + BP + CP = s$;

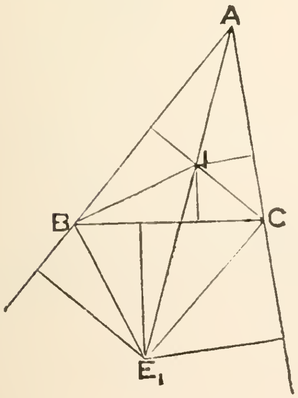
$$\therefore \quad AR = s - a = AQ.$$

Similarly $CP = s - c = BX$,

&c.

Cor. (1) $PX = CX \sim CP = (s-b) \sim (s-c) = c \sim b = AB \sim AC$.
Cor. (2) $RZ = AZ - AR = s - (s-a) = a = BC$.

THEOREM (5)—*If* r *is equal to the radius of the in-circle, and* r_1, r_2, r_3 *are respectively equal to the radii of the ex-circles relatively to* a, b, c, *then the area of the triangle* ABC *is equal to any of the rectangles* rs, $r_1(s-a)$, $r_2(s-b)$, $r_3(s-c)$.



Let I be centre of in-⊙; and E_1 of ex-⊙ relatively to BC.

Then area ABC

= area BIC + area CIA + area AIB,

$= \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc$,

$= rs$.

Again, area ABC

= area AE_1B + area AE_1C − area BE_1C,

$= \frac{1}{2}r_1c + \frac{1}{2}r_1b - \frac{1}{2}r_1a$,

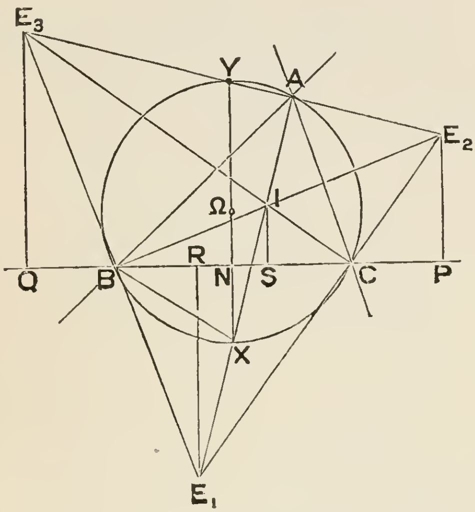
$= r_1(s-a)$.

Similarly

$= r_2(s-b) = r_3(s-c)$.

SOME USEFUL THEOREMS, MAINLY DEPENDING ON BOOK IV.

THEOREM (6)—*If the in-centre and three ex-centres are joined, the six joins are bisected by the circum-circle.*



Let I be in-centre of △ ABC, and E_1, E_2, E_3, the ex-centres of ⊙s touching BC, CA, AB respectively.

Then AIE_1 and E_2AE_3 being int. and ext. bisectors of $B\hat{A}C$, cut circum-⊙ at X, Y so that XY is diam. bisecting BC at rt. ∧s.

But if P, Q are pts. of contact of ⊙s (centres E_2 E_3) with BC produced, E_2P, E_3Q are ⊥ to BC.

And BQ = s − a = CP.

∴ XY bisects PQ.

And being ‖ to E_2P, E_3Q, it also bisects E_2E_3.

Similarly circum-⊙ bisects E_1E_2 and E_3E_1.

Again, if R, S are pts. of contact of ⊙s (centres E_1, I) with BC.

BR = s − c = CS,

∴ XY bisects RS.

And being ‖ to IS, E_1R, it also bisects IE_1.

Similarly circum-⊙ bisects IE_2 and IE_3.

Cor. (1). The circum-⊙ of ABC is the N.P. ⊙ of $E_1E_2E_3$.

Cor. (2). The in-centre of ABC is the ortho-centre of $E_1E_2E_3$; and ABC is the pedal △ of $E_1E_2E_3$.

Cor. (3). Since $X\hat{I}B = \frac{1}{2}(B\hat{A}C + A\hat{B}C) = X\hat{B}I$, and $I\hat{B}E_1$ is rt.

∴ $XI = XB = XE_1$.

THEOREM (7)—*If* R *is the radius of the circum-circle, then*

$$r_1 + r_2 + r_3 - r = 4R.$$

For if XY (last fig.) cuts BC in N,
since Y is mid pt. of $E_2 E_3$,
and $E_2 P$, YN, $E_3 Q$ are $\perp$ to PQ,
$\therefore \quad 2\,YN = r_2 + r_3.$
Similarly $\quad 2\,XN = r_1 - r.$
$\therefore \quad 4R$ (which $= 2\,YN + 2\,XN$) $= r_1 + r_2 + r_3 - r.$

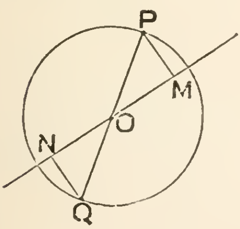
Cor. (1). $\Sigma(XN) = \frac{1}{2}(r_1 + r_2 + r_3 - 3r) = 2R - r.$

Cor. (2). If Ω is the circum-centre,

$$\Sigma(\Omega N) = 3R - \Sigma(XN) = R + r.$$

Note—When $B\hat{A}C$ is obtuse, X and N lie on opposite sides of Ω, and then $R = XN - \Omega N$. The formula $\Sigma(\Omega N) = R + r$, will include all cases, if we make the convention that ΩN is to be considered *negative* when Ω and X are on the *same side* of BC.

THEOREM (8)—*If* A, B, C, &c. *are the corners of a regular polygon, and* O *the common centre of its in- and circum-circles, then* O *is the mean centre of the points when each of the multiples is equal to unity.*

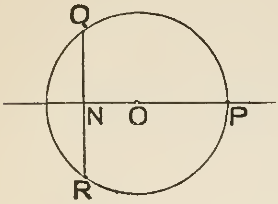


1°, if the pol. has an *even* number of sides.

Every pt. P has another pt. Q, at the opposite end of the diam. POQ.

And if *any* line is drawn thro. O, and $\perp^s$ PM, QN dropped on it, PM, QN are clearly equal in length, and opposite in direction.

$\therefore$ for all such points
$\Sigma(PM) = 0.$
$\therefore$ O is their mean centre.



2°, if pol. has an *odd* number of sides.

Then for any pt. P all the other pts. are situated in pairs, as Q, R, on opposite sides of, and equidistant from diam. PO.

So that if QR cuts PO in N,

QNR is bisected by, and $\perp$ to PO.

$\therefore\ \Sigma(\mathrm{QN}) = 0$, for that line.

$\therefore$ mean centre is in PO.

Similarly for every diam. thro. a corner of the pol.

$\therefore$ O is the mean centre of the corners.

THEOREM (9)—*If* O *is the centre, and* OR *the radius of the circle circumscribing a regular polygon* ABCD &c. *of* n *sides; then for every point* P

$$\Sigma(\mathrm{AP}^2) = n(\mathrm{OR}^2 + \mathrm{OP}^2)$$

$$\textit{and}\quad \Sigma(\mathrm{AB}^2) = n^2 . \mathrm{OR}^2$$

where $\Sigma(\mathrm{AB}^2)$ *includes every binary combination of the* n *letters* A, B, C, &c.

For O is the mean centre of the corners of the pol. when each of the mults. = 1.

$\therefore$, by ii. *Addenda* (20), *Cor.*,

$$\Sigma(\mathrm{AP}^2) = \Sigma(\mathrm{AO}^2) + n . \mathrm{OP}^2.$$

But OA = OB = OC = &c. = OR.

$$\therefore\ \Sigma(\mathrm{AP}^2) = n . \mathrm{OR}^2 + n . \mathrm{OP}^2.$$

Next, in the particular case when P is on $\odot$, OP = OR.

And then $\Sigma(\mathrm{AP}^2) = 2n . \mathrm{OR}^2$.

Now suppose P to coincide successively with A, B, C, D, &c.

$$\text{Then}\quad \mathrm{AA}^2 + \mathrm{AB}^2 + \mathrm{AC}^2 + \mathrm{AD}^2 + \&c. = 2n . \mathrm{OR}^2,$$

$$\mathrm{BA}^2 + \mathrm{BB}^2 + \mathrm{BC}^2 + \mathrm{BD}^2 + \&c. = 2n . \mathrm{OR}^2,$$

$$\mathrm{CA}^2 + \mathrm{CB}^2 + \mathrm{CC}^2 + \mathrm{CD}^2 + \&c. = 2n . \mathrm{OR}^2,$$

&c.

Adding, and recollecting that AA = 0, BB = 0, &c., we get

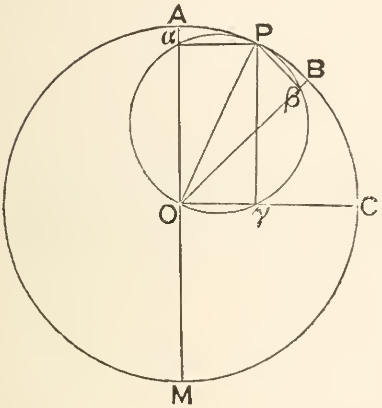
$$2\Sigma(\mathrm{AB}^2) = n . 2n . \mathrm{OR}^2,$$

$$\text{or}\quad \Sigma(\mathrm{AB}^2) = n^2 . \mathrm{OR}^2.$$

THEOREM (10)—*If* P *is any point on the circum-circle of a regular polygon* ABCD &c. *of* n *sides;* L *any line through* O *the centre, and* $P\alpha$, $P\beta$, &c., *the respective perpendiculars on* OA, OB, &c.; *and* AL, BL, &c., *the perdiculars from* A, B, &c., *on* L; *then—*R *being the radius—*

$$\Sigma(P\alpha^2) = \frac{n}{2} \cdot R^2 = \Sigma(O\alpha^2);$$

$$\textit{and } \Sigma(AL^2) = \frac{n}{2} \cdot R^2 = \Sigma(OL^2).$$



With OP as diam. describe a ⊙, then α, β, γ, &c. will all lie on this ⊙.

1°, if n is *odd*, there will be n distinct pts. α, β, γ, &c.

And, since $\alpha \hat{O} \beta$, $\beta \hat{O} \gamma$, &c., each $= \frac{1}{n}$th of 4 rt. $\wedge^s$.

$\therefore$ pol. $\alpha\beta\gamma$ &c. is regr.

$$\therefore \text{ by Theor. (9) } \Sigma(P\alpha^2) = 2n\left(\frac{OP}{2}\right)^2 = \frac{n}{2} \cdot R^2.$$

2°, if n is *even*, then for each pt. A there will be another pt. (say M) at opposite end of diam. thro. A, so that the $\perp^s$ from P on OA and OM coincide;

i.e. α and μ are one pt.

$\therefore$ $\alpha\beta\gamma$ &c. will form a regr. pol. of $\frac{n}{2}$ sides.

And sum of sqs. of $\perp^s$ from P on *all* radii drawn to A, B, C, &c. = twice sum of sqs. of joins of P to corners of pol. $\alpha\beta\gamma$ &c.

$$= 2 \times 2\left(\frac{n}{2}\right)\left(\frac{OP}{2}\right)^2, \text{ by Theor. (9)}$$

$$= \frac{n}{2} \cdot R^2, \text{ as before.}$$

And, by precisely similar reasoning,

$$\Sigma(\mathrm{O}\alpha^2) = \frac{n}{2} \, . \, \mathrm{R}^2.$$

Again, since for any two pts. on a ⊙,

⊥ from 1st on diam. through 2nd = ⊥ from 2nd on diam. through 1st.

∴ each ⊥ from n pts. A, B, C, &c. on diam. thro. any other pt. P = corresponding ⊥ from P on diam. thro. A, B, C, &c.

$$\therefore \quad \Sigma(\mathrm{AL}^2) = \Sigma(\mathrm{P}\alpha^2) = \frac{n}{2} \, . \, \mathrm{R}^2.$$

$$\text{Similarly} \quad \Sigma(\mathrm{OL}^2) = \Sigma(\mathrm{O}\alpha^2) = \frac{n}{2} \, . \, \mathrm{R}^2.$$

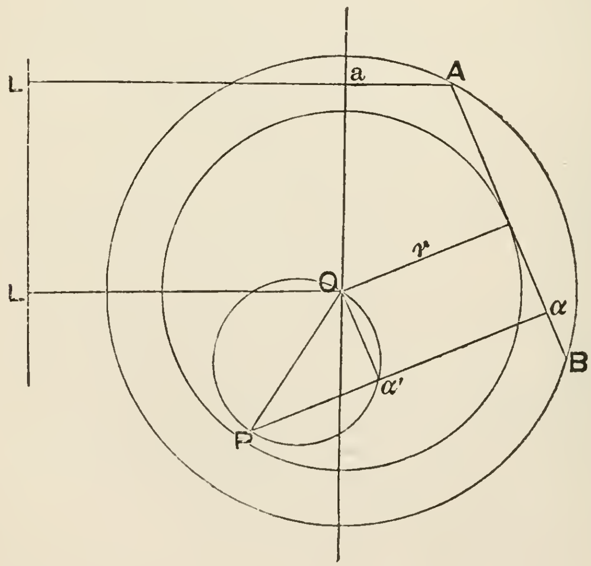
THEOREM (11)—*If* O *is the common centre of the in- and circum-circles of a regular polygon* ABCD &c. *of* n *sides; and if* P *is any point, and* L *any straight line; then, if* Pα, Pβ, &c. *are the perpendiculars from* P *on* AB, BC, &c., *and* AL, BL, &c. *the perpendiculars from* A, B, C, &c. *on* L,

$$\Sigma(\mathrm{P}\alpha^2) = n\,(r^2 + \tfrac{1}{2}\,\mathrm{OP}^2),$$

$$\textit{and} \quad \Sigma(\mathrm{AL}^2) = n\,(\mathrm{OL}^2 + \tfrac{1}{2}\,\mathrm{R}^2),$$

where r = *radius of in-circle,*

and R = „ *circum-circle.*



Lemma. $\Sigma(\mathsf{P}\alpha) = \mathrm{n\,r}$.

$$\text{For } \mathsf{P}\alpha \,.\, \mathsf{AB} + \mathsf{P}\beta \,.\, \mathsf{BC} + \&c.$$
$$= \text{twice area pol.}$$
$$= \mathrm{r} \,.\, \mathsf{AB} + \mathrm{r} \,.\, \mathsf{BC} + \&c.$$
$$\therefore \quad \mathsf{AB}\,\Sigma(\mathsf{P}\alpha) = \mathrm{n\,r} \,.\, \mathsf{AB},$$

whence Lemma is true.

Let α', β', γ', &c. be feet of $\perp^s$ from O on $\mathsf{P}\alpha$, $\mathsf{P}\beta$, $\mathsf{P}\gamma$, &c. respectively.

Then α', β', γ', &c. lie on $\odot$ whose diam. is OP.

$$\text{And } \Sigma(\mathsf{P}\alpha') = \Sigma(\mathsf{P}\alpha - \mathrm{r}) = \Sigma(\mathsf{P}\alpha) - \mathrm{nr} = 0.$$
$$\text{Now } \mathsf{P}\alpha = \mathrm{r} + \mathsf{P}\alpha',$$
$$\therefore \quad \mathsf{P}\alpha^2 = \mathrm{r}^2 + \mathsf{P}\alpha'^2 + 2\,\mathrm{r}\,\mathsf{P}\alpha',$$
$$\therefore \quad \Sigma(\mathsf{P}\alpha^2) = \mathrm{n\,r}^2 + \Sigma(\mathsf{P}\alpha'^2) + 2\,\mathrm{r}\,\Sigma(\mathsf{P}\alpha'),$$
$$= \mathrm{n\,r}^2 + \frac{\mathrm{n}}{2}\,\mathsf{OP}^2.$$

Next let diam. thro. O, ∥ to L, cut AL, BL, CL, &c. in a, b, c, &c. respectively.

$$\text{Then} \quad \mathsf{AL} = \mathsf{Aa} + \mathsf{OL},$$
$$\therefore \quad \mathsf{AL}^2 = \mathsf{Aa}^2 + \mathsf{OL}^2 + 2\,\mathsf{OL} \,.\, \mathsf{Aa},$$
$$\therefore \quad \Sigma(\mathsf{AL}^2) = \Sigma(\mathsf{Aa}^2) + \mathrm{n} \,.\, \mathsf{OL}^2 + 2\,\mathsf{OL} \,.\, \Sigma(\mathsf{Aa}).$$
$$\text{But} \quad \Sigma(\mathsf{Aa}) = \Sigma(\mathsf{AL}) - \mathrm{n} \,.\, \mathsf{OL},$$
$$= 0, \because \text{ O is mean centre of A, B, C, \&c.}$$
$$\therefore \quad \Sigma(\mathsf{AL}^2) = \frac{\mathrm{n}}{2}\,\mathsf{R}^2 + \mathrm{n} \,.\, \mathsf{OL}^2.$$

Cor. If P is on the circumf. of the in-$\odot$, and L is a tang. to the same, then—

1°, if $\perp^s$ are dropped on sides of a regr. pol. of n sides, from any pt. on its in-$\odot$,

$$\text{sum of sqs. on these } \perp^s = \tfrac{3}{2}\,\mathrm{n}\,(\text{rad. }\odot)^2.$$

2°, if $\perp^s$ are dropped from corners of a regr. pol. of n sides on any tang. to its circum-$\odot$,

$$\text{sum of sqs. on these } \perp^s = \tfrac{3}{2}\,\mathrm{n}\,(\text{rad. }\odot)^2.$$

Note—Hence, if a point is subject only to the *condition* that the sum of the squares on its distances from the corners or sides of a regular polygon is constant, its *locus* is one of a series of circles, concentric with the in- and circum-circles of the polygon—the particular value of the constant determining the particular circle which is the locus.

EXERCISES ON BOOK iv.

NOTE—*The following are Theorems to be proved; and depend mainly on the principles of Book* iv.

1. If the in- and circum-centres of a triangle coincide, the triangle is equilateral.

2. If the join of the in- and circum-centres goes through a corner of the triangle, it is isosceles.

3. If one square is inscribed in another, the difference of their areas is equal to twice the rectangle under the segments of a side of the outer made by a corner of the inner.

4. The join of the centres of the in- and circum-circles of a triangle subtends at any corner an angle which is half the difference of the angles at the other corners.

NOTE—*If* I *is in-centre*, S *circum-centre*, *of* △ ABC ; *then*

$$A\hat{B}C \sim A\hat{C}B = S\hat{B}A \sim S\hat{C}A = S\hat{A}B \sim S\hat{A}C = 2\,S\hat{A}I$$

a useful result, which should be known.

5. The circles, each of which touches two sides of a regular pentagon at the extremities of a third, all go through the common centre of the in- and circum-circles of the pentagon.

6. If ABCDE is *any* pentagon inscribed in a circle, and AC, BD, CE, DA, EB are joined; then

$$A\hat{B}E + B\hat{C}A + C\hat{D}B + D\hat{E}C + E\hat{A}D = 2 \text{ rt. } \wedge^{s}.$$

7. If two equilateral triangles circumscribe a circle, their intersections will form a hexagon which is always equilateral, but *not* always equiangular.

8. If two sides of a triangle, whose perimeter is constant, are given in position, then the third side always touches a fixed circle.

NOTE—*Use* iv. *Addenda* (4).

9. The square on a side of an equilateral triangle inscribed in a circle is three times the square on a side of a regular hexagon inscribed in the same circle.

10. The area of a regular hexagon inscribed in a circle is three-fourths the regular hexagon circumscribed about the circle.

11. If two diagonals of a regular pentagon intersect, the larger segment of each is equal to a side of the pentagon.

12. The difference between a side and diagonal of a regular pentagon is equal to a side of another regular pentagon, whose diagonal is a side of the first.

13. A regular octagon is equal to the rectangle under a side of the square inscribed in, and a side of the square circumscribed about the circle which circumscribes the octagon.

14. If the radius of a circle is cut in *medial section*, the greater segment is equal to a side of a regular decagon inscribed in the circle.

15. AB, BC, CD are three adjacent sides of a regular polygon; if AB, DC are produced to meet in X, then A, X, C are concyclic with the centre of the circle round the polygon.

16. The square on the diameter of the circum-circle of a regular pentagon is equal to the square on one of its sides together with the square on the diameter of the in-circle.

17. If any hexagon is inscribed in a circle, the sum of either set of its alternate angles is equal to four right angles.

18. If the extremities of a side of a regular pentagon, inscribed in a circle (radius r) are joined to the mid point of the minor arc subtended by either side adjacent to it; then—

1°, the difference of the joins $=$ r;

2°, the sum of the squares on the joins $= 3\,\mathrm{r}^2$;

3°, the rectangle under the joins $= \mathrm{r}^2$.

NOTE—*In the fig. of* iv. 10 *produce* CP *to meet* ⊙ *in* Q: *then* QB *is side of inscribed regular pentagon, and* C *is the mid pt. of an adjacent arc, so that* BC, QC *are the 'joins.'*

19. The mid points of the sides of a square are joined, forming an inscribed square; and again the mid points of this new square, forming another inscribed square; and this process is continued: then the limit of the sum of all the inscribed squares is the area of the original square.

NOTE—*Recollect that—Limit of* $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. ad. infin $= 1$.

20. The area of each of the triangles made by joining the extremities of adjoining sides of a regular pentagon is less than one-third, but greater than one-fourth of the area of the pentagon.

21. ABCDE is a regular pentagon, and AC, BD cut in O; then

$$AC \,.\, CO = BC^2.$$

22. If a ribbon is folded into a flat five-sided knot, the knot forms a regular pentagon.

NOTE—*The edges of the ribbon give 4 diags., and these are* ∥ *to sides, and form 3 isos.* △s, *each of whose base* ∧s *= ext.* ∧ *of pent. Use* Cor. iii. 26, (e).

23. Two triangles PQR, XYZ are so inscribed in a circle that PX, QY, RZ are concurrent in O; *if* O is the in-centre of one triangle then O is also the orthocentre of the other; and conversely.

24. If R, r are the respective radii of the circum-circle, and in-circle, of a regular polygon; and R′, r′ corresponding quantities for the regular polygon of same perimeter, but double the number of sides; then—

$$R + r = 2r', \text{ and } Rr' = R'^2.$$

NOTE—*If* AB *is a side of first pol.;* M *mid pt. of arc* AB*;* X, Y *mid pts. of* AM, BM; *then* XY *is a side of second pol.*

25. If two circles cut orthogonally, and the distance between their centres is twice the radius of one of them; then their common chord is a side of a regular hexagon inscribed in one of them, and of an equilateral triangle in the other.

NOTE—*Circles are said to* cut orthogonally, *when the tangent to one, at a common point, is a normal to the other.*

26. If ABCP and A′B′C′P′ are concentric circles; and ABC, A′B′C′ equilateral triangles in them; then—

$$AP'^2 + BP'^2 + CP'^2 = A'P^2 + B'P^2 + C'P^2.$$

NOTE—*Use* ii. *Addenda* (16) *Cor.*

27. If the in-circle of a triangle ABC, touches BC in D, then the in-circles of triangles ABD, ACD will touch each other.

NOTE—*Use* iv. *Addenda* (4).

28. If circles are inscribed in the two triangles into which a triangle is divided by an altitude, and analogous circles are drawn in relation to the two other altitudes; then—

sum of diams. 6 $\odot^s$ + sum sides of original $\triangle$ = 2 sum of altitudes.

NOTE—*Consider one pair of the* $\triangle^s$, *and use* iv. *Addenda* (4). *Do* NOT *draw the* $\odot^s$.

29. The square on a side of a regular pentagon in a circle is equal to the square on a side of a regular hexagon in same circle, together with the square on a side of a regular decagon in the circle.

NOTE—In fig. iv. 10, *produce* CP *to* Q: *then* BC *is a side of the decagon,* BQ *of the pentagon, and* AB *of the hexagon: also* QP = QA; *and result follows from* ii. *Addenda* (9).

30. AB, CD are perpendicular diameters of a circle (centre O), OB is bisected in X, and Y taken in XA so that XY is equal to XC: then CY is equal to a side of the inscribed pentagon.

NOTE—*By* ii. 11, *show that* AO . AY = OY²*: then, by* iv. 10, YO = *side of decagon in the* $\odot$; *and result comes from last Exercise.*

PROBLEMS.

None have been hitherto given in the sets of Exercises, because beginners in Geometry invariably find the solution of Problems altogether hopeless. The reason of this seems to be that the solution of a Problem—to which the solver has no previous clue—is, so far as he himself is concerned, original work; whereas the demonstration of a Theorem is merely a retracing of steps to some extent indicated.

There are three ways of achieving the solution of a Problem—

1°, by a felicitous guess, which can rarely be made, excepting in very simple cases.

2°, when the *data* are such that, one of them being omitted, the others give a Locus; then the intersection of the Loci will give the solution.

For example—given base, vertical angle, and area of a triangle, to construct it.

Omitting the area, we get Locus (ζ) p. 178.

Omitting the vertical angle, we get Locus (ι) p. 178.

The intersection of these Loci gives two triangles which satisfy the *data*.

3°, by what is called the *analysis* of the Problem—a process which will be most readily understood by seeing a teacher employ it. The essentials of the method consist in assuming the Problem solved, and then endeavouring to find some connection between the *quæsita* and the *data*: hence a clue may often be found to the putting together (*synthesis* as it is called) of the figure.

As an example, take ii. 11. We may imagine the construction found out thus—

Assume X such a pt. in the given line AB that $AX^2 = AB \cdot BX$:

i.e. that sq. AXGF, on AX, = rect. XBCH, in which BC = BA.

Then it would readily occur to one that, if rect. AH was completed, AC would be sq. on AB.

So that, adding AH to the (assumed) equals, we should get

$$AB^2 = DF \cdot FG = DF \cdot FA.$$

Here might be imagined a pause in the process of discovery; until, in a lucky moment, the idea of applying ii. 6 to DF (considered as DA produced) would suggest adding sq. on half AD to each side; after which the solution would become obvious.

NOTE—*The following are solvable by Book* i.

1. Construct an isosceles triangle so that each side is double the base.

2. Construct an isosceles triangle when the lengths of its base and of its equal sides are given.

3. Construct a square when its diagonal is given.

4. Construct a triangle with an angle equal to a given angle, and with two sides (one of which is opposite that angle) of given lengths.

When is the solution of the Problem impossible; and when ambiguous?

5. Given the base and area of a triangle, find the position of its vertex.

6. In the base of a triangle find the point from which parallels to the sides, terminated by them, are equal.

7. Construct a parallelogram when the lengths of one side and of the two diagonals are given.

8. Given two points and a line, describe a circle which shall pass through the points and have its centre on the line.

9. In a given triangle place a line, so as to be of given length, parallel to a given direction, and terminated by the sides of the triangle.

10. OA, OB are lines fixed in direction; and P is a given point in OA: find X, in OA, so that XP may be equal to the distance of X from OB.

11. Given two squares; describe another square, so that its corners may be on the sides of one of them, and its area equal to that of the other.

When is the solution of the Problem impossible?

12. Trisect a right angle.

13. Divide a given line into any given number of equal parts.

14. Find a construction to trisect a given line, which will also solve the Problem—To divide an equilateral triangle into nine parts of equal area (p. 85).

15. From a given point draw three lines, of given lengths, so that their extremities may be in one line, and one of them equidistant from the others.

NOTE—*Describe a* $\triangle$, *so that the given point may be one of its corners; that two of its sides may be two of the given lengths; and that the third side may be double the third length.*

16. Given a line XY and two points A, B; find a point P in XY such that PA, PB will make equal angles with XY.

NOTE—*Draw* AN $\perp$ *to* XY, *and produce to* A′, *so that* A′N = AN: *then* A′B *will cut* XY *in* P.

17. Given two intersecting lines, and a point between them; it is required to draw a line through the point, and terminated by the lines, so that the point may be its mid point.

18. Inscribe a parallelogram in a given triangle, so that its diagonals may intersect at a given point within the triangle.

19. AOB is a given angle, and P a given point either within or without it; through P draw a line to cut OA in X, and OB in Y, so that PX may be m times PY; where m is a given whole number.

20. Given two lines, which are not parallel, find, without producing them to meet, the line which would bisect the angle between them.

NOTE—*Draw any two lines across the given lines, and use the property of* i. *Addenda* (24).

21. Draw a line of given length, parallel to a given direction, and terminated by the circumferences of two given circles.

22. Draw from a corner of a triangle a line to cut off from the triangle a given area.

23. Bisect a triangle by a line through a given point in one of its sides.

NOTE—*For this and the next two Probs., see p.* 85.

24. Find a construction that will quadrisect any quadrilateral.

25. Construct an equilateral triangle, so that its corners may be at given distances from a given point.

26. Bisect a quadrilateral by a line through one of its corners.

NOTE—*See p.* 80, *Exercise* 44.

27. Construct a triangle, when given—

1°, a median and two sides;

2°, a side and two medians.

28. Construct a triangle, when given its base, an angle at the base, and—

1°, the *sum* of its other two sides;

2°, „ *diff.* „ „

29. Construct a triangle, when given an altitude, an angle at the base, and the sum of its three sides.

30. Bisect a rectangle by two lines parallel to two of its adjacent sides, and equidistant from them.

NOTE—*If* ABCD *is the rect., draw* ||s *from the in-centre of* △ ABC.

31. Given two circles and a point; draw a line, terminated by the circles, so that the point may be its mid point.

Find the conditions under which a solution is possible.

32. Find a construction to alter the shape of a rectilineal figure so that its area may remain the same, but that the new figure may have one side less than the old one.

33. Inscribe a square in a given semi-circle.

NOTE—*The solutions of the following depend mainly on Book* ii.

1. Produce a given line AB to X, so that the rectangle under AX + AB and AX – AB may be equal to a given square.

2. Divide a given line into two parts, so that the difference of the squares on them may be equal to a given square.

3. Divide a given line into two parts, so that the sum of the squares on them may be equal to a given square.

4. Produce a given line, so that the difference of the squares on the whole line produced and on the part produced may be equal to a given square.

5. Produce a given line, so that the sum of the squares on the whole line produced and on the part produced may be equal to a given square.

6. Divide a given line into two parts, so that the square on one of them may be double the square on the other.

7. Produce a given line, so that the square on the whole line produced may be double the square on the part produced.

8. Divide a given line, 1°, *internally*, 2°, *externally*, so that the sum of the squares on the original line and on one segment may be treble the square on the other segment.

NOTE—*The pts. of internal and external medial section solve this Prob., and the following.*

9. Divide a given line, 1°, *internally*, 2°, *externally*, so that the square on the line made up of the original line and one segment may be quintuple the square on the other segment.

10. Produce AB to X, so that $AX \cdot BX = AB^2$.

11. Produce AB to X, so that $AB^2 + BX^2 = 2\,AX \cdot BX$.

12. Produce AB to X, so that AX^2, 1°, $= 3\,AB^2$; and, 2°, $= 5\,AB^2$.

13. Divide AB in X, so that $AB^2 + BX^2 = 2\,AX^2$.

NOTE—*The analysis of the Prob. will lead to this construction: produce given line* AB *to* Y *so that* $AY^2 = 3\,AB^2$; *take* X, *in* AB, *so that* AX = BY.

14. Divide a given line into two parts, so that the rectangle under them may be equal to the square on their difference.

NOTE—*In the fig. of* ii. 14, *if* FY *is* ⊥ *and equal to* FB, *and* YO *cuts* ⊙ *in* X, *then* XE *divides given line* BF *as reqd.*

15. Produce a given line so that the rectangle under the whole line produced and the original line may be equal to a given square.

16. ABC is an isosceles triangle: find P, in the base BC, so that, if the perpendicular to BC at P meets AB in X, then—

$$PA^2 + PX^2 = AB^2.$$

NOTE—*The following require the aid of* Book iii.

1. Through one given point draw the line which passes at a given distance from another given point.

2. Find a point outside a given circle from which if tangents are drawn they will contain a given angle.

3. Through two given points describe a circle bisecting the circumference of a given circle.

4. Draw a line to touch a given circle, and make a given angle with a given line.

5. In the production of a diameter of a circle find the point from which a tangent is equal to a diameter.

6. Describe two circles, of given radii, so as to touch a given line and each other.

7. Given two intersecting circles, draw through one of their points of section a line so as to be terminated by the circumferences and bisected at the point.

8. Draw PXYQ across two con-centric circles, so that (P, Q being on the outer; and X, Y on the inner) 1°, PQ = 2 XY; and, 2°, PQ = 3 XY.

9. If A is a given point, and C the centre of a given circle; find P in CA so that PA is equal to the tangent from P.

10. Through a point within a circle draw a chord so that the rectangle under the whole chord and one part is equal to a given square.

What are the limits to the side of this square?

11. Two circles have internal contact at P: draw PXY to meet them in X, Y, so that XY may be of given length.

12. AB is a fixed chord, and AX a variable chord of the same circle: find the Locus of the intersection of the diagonals of the parallelogram of which AB, AX are adjacent sides; find also when the diagonal from A has its greatest length.

13. Any line is drawn across a triangle ABC, and meets the sides (produced if necessary) opposite A, B, C, in X, Y, Z respectively: find the Locus of the other point of section of the circumcircles of triangles CXY, AYZ.

14. Through a given point P, within a given angle AOB, draw a line to meet OA, OB, in X, Y, so that the rectangle under XP, PY may be equal to a given square.

15. Draw a circle to cut three given non-intersecting circles orthogonally.

NOTE—*Draw an ext. common tang.* (cf. p. 226, 4) *to one pair of* ⊙s, *and to another pair of* ⊙s: *the* ⊥s *to the respective lines of centres, bisecting these common tangs. intersect in the centre of the orthogonal* ⊙.

NOTE—*Here follow some Problems solvable by the methods of Books* i-iv.

1. Inscribe an equilateral triangle in a square, so that a corner may be—

 1°, at mid point of a side;

 2°, coincident with a corner;

 3°, at any given point on a side.

NOTE—*Cases* 1° *and* 2° *present no difficulty: for* 3°, *let* ABCD *be sq.;* P *given pt. in* CD, *nearer to* C *than* D: *let* ⊥ *to* PD, *at its mid pt. meet trisector of* $\hat{A}$, *nearest* AD, *in* X: *then* PX *produced is side of* △.

2. Describe a square, when—

 1°, its area = *sum* of areas of two given squares;

 2°, „ = *diff.* „ „

 3°, the *sum* of a side and diagonal is given;

 4°, *diff.* „ „

3. Draw a line parallel to one side of a triangle (considered as base) and terminated by the other two, so that its length may be equal to the—

 1°, *sum* of the segments cut off between it and the base;

 2°, *diff.* „ „ „

4. Draw all the common tangents to two given non-intersecting circles.

NOTE—*With centre of larger* ⊙, *and radius which is* 1°, *the diff.*, 2°, *the sum of the given radii, describe another* ⊙; *and draw a tang. to it from the centre of the smaller* ⊙.

5. Find the Loci of *all* the points of intersection of common tangents to two circles, whose centres are fixed, but radii variable.

6. A and B are given circles, on same side of a given line XY: find a point in XY such that tangents from it to A and B will make equal angles with XY.

NOTE—*Four such pts. can generally be found.*

7. Construct a triangle, when given its base, vertical angle, and—

 1°, *sum* of the other two sides;

 2°, *diff.* „ „

8. Construct a triangle, when given its base, difference of base angles, and—

 1°, *sum* of the other two sides;

 2°, *diff.* „ „

9. Construct a triangle, when given the lengths of its three medians.

NOTE—*In the fig. of* i. *Addenda* (20) *the sides of* △ GβH *are each* $\frac{2}{3}$ *of a median.*

10. Examine what Problems could be solved by taking the Loci on pp. 177 178, two and two together.

11. Find two lines, when given any two of the following—

(α) their sum:

(β) their difference:

(γ) the rectangle under them:

(δ) the sum of the squares on them:

(ϵ) the difference of the squares on them.

NOTE—*The difficult case is when* (γ) *and* (ϵ) *are taken together: its consideration should be deferred until Book* vi *has been read.*

12. Given two lines, it is required to produce one of them so that the rectangle under the whole line thus produced, and the other given line, may be equal to the square on the produced part.

NOTE—AB *line to be produced;* BC *other line, placed so that* ABC *is a line. On* AC *place semi-*⊙; *and let* BP, ⊥ *to* ABC, *meet it in* P. *Join* P *to* M, *the mid pt. of* BC. *Produce* BC *to* X, *so that* MX = MP. *Then* BX *is the reqd. production of* AB.

13. Divide a line; 1°, internally; 2°, externally; so that the rectangle under the segments may be equal to a given area.

14. Produce a given line both ways, so that the rectangles under the segments into which the whole produced line is divided at the extremities of the given line may be equal to given areas.

15. Describe a circle to bisect the circumferences of three given circles.

NOTE—*Apply the last Exercise to the joins of the three given centres.*

16. To divide a given line into two parts, so that the square on one of them may be equal to the rectangle under the other and another given line.

NOTE—*Place the given lines* AB, AC *so that* BAC *is one line: on same side of* AC, BC *place semi-*⊙s: *let tang. at* A *to one, meet the other in* P: *let join of* P *with cent. of lesser* ⊙ *meet it in* Q: *tang. at* Q *cuts* AB *in the required pt.*

17. To divide a given line into two parts, so that the rectangle under the whole line and one part may be m times the square on the other.

NOTE—*Produce the given line so that the produced part* = $\frac{1}{m}$*th of itself; and construct as in the last Prob.*

18. Construct a triangle when given the lengths of the bisector of an angle, and of the median and altitude concurrent with that bisector.

NOTE—*In fig. of* i. *Addenda* (26) *let* C *be circum-centre; then* CD, aA *meet on circum-*⊙; *whence* aA *bisects* $\widehat{CaX}$.

19. Construct a triangle when given the lengths of the bisector of an angle, of the altitude concurrent with it, and of the radius of the circum-circle.

20. Construct a triangle when given two of its angles, and its perimeter.

21. Through a common point of two intersecting circles draw a double chord of given length.

22. Given two triangles, circumscribe a triangle about the lesser of them so as to be identically equal to the greater.

NOTE—*Use the preceding Problem.*

23. Given three parallel lines, construct an equilateral triangle with its corners on them.

24. Construct a triangle when given an angle, the altitude from the vertex of that angle, and the radius of the in-circle.

25. From a given point draw a line to form with a given angle a triangle of given perimeter.

NOTE—*See* iv. *Addenda* (4).

26. Through a given point in a diameter of a circle draw a chord so that of the two arcs intercepted between the chord and diameter one may be three times the other.

27. From a point of intersection of two circles draw a line to cut them, so that the rectangle under the parts of it, which are chords of the circles, may be given.

28. Construct a triangle, when given the rectangle under the sides of an angle, the median to the opposite side, and the difference of the other angles.

NOTE—*See* iii. *Addenda* (15); i. *Addenda* (10), (11); iii. *Ex.* 61; *and use the preceding Problem.*

29. Construct a triangle, when given an angle, the altitude drawn from the corner of it, and—

1°, the *sum* of the sides containing it;

2°, the *diff.* " "

30. Construct a triangle, when given its base, difference of base angles, and that the Locus of its vertex is—

1°, a line parallel to the base;

2°, a line cutting the base.

31. Inscribe in a given circle a polygon of n sides, so that one side may go through a given point, and the other sides may be parallel to given directions.

NOTE—*Consider separately the cases when* n *is odd, and when even. Draw lines* || *to the given directions, beginning at any pt. on the* ⊙.

BOOK V.

An abridgment of Euclid's Fifth Book—mainly based on *De Morgan's Connexion of Number and Magnitude.*

By the word *number*, in what follows, we merely convey the notion of times or repetitions, considered independently of the things counted or repeated.

By the word *magnitude* is meant a thing presented to us simply as that which is made up of parts, not differing from the whole in anything but in being less; so that, if we consider separately a part and the whole, we have only two inferences—

The part is less than the whole,

The whole is greater than the part.

We shall use capital letters, as A, B, C, &c., to represent magnitudes—not as in algebra, the number of units which the magnitudes contain, but the magnitudes themselves—so that if it be, for example, weight of which we are speaking, A is not a number of pounds, but the weight itself.

Concerning magnitudes we shall only assume that magnitudes of the same kind may be added together, or that the same magnitude may be repeated any number of times; and that a lesser magnitude may be taken from another of the same kind.

We shall use small letters as $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p}$, &c., to denote integer numbers as just defined; and any one of them placed before a capital will denote repetition of the magnitude represented by that capital: thus as 3 A denotes A repeated thrice, so $\mathbf{m}$ A denotes A repeated $\mathbf{m}$ times.

Def. When a greater magnitude contains a lesser magnitude a definite number of times *exactly*, the greater magnitude is called a *multiple* of the lesser; and the lesser is called a *sub-multiple* of the greater.

If the greater magnitude is denoted by A, and the lesser by B, then the relation expressed in this definition will be denoted thus—

$$A = \mathbf{m}\,B,$$

which is to be considered merely as an equivalent for these words—

'A is a multiple of B.'

So we might have $B = \mathbf{n}\,C$, $C = \mathbf{p}\,D$, &c., with similar significations.

Def. A sub-multiple is sometimes said *to measure*, or be *a measure* of its multiple.

Hence the word *commensurable*, which means *having a common measure;* so also *incommensurable* means not *having a common measure.*

We shall assume that the following properties of multiples are evident—

1. $A >, =, \text{ or } < B$, according as $mA >, =, \text{ or } < mB$.
   And conversely.
2. $mA + mB = m(A + B)$.
3. $mA - mB = m(A - B)$, A being greater than B.
4. $mA + nA = (m + n)A$.
5. $mA - nA = (m - n)A$, m being greater than n.
6. $m.nA = mn.A = nm.A = n.mA$.

From the mutual relationship of two magnitudes of the same kind, there arises a *tertium quid*, which represents their *relative*, as distinguished from their *absolute* greatness, and which is called the *ratio* of the magnitudes.

In the arithmetical treatment of magnitude, where it is assumed that all magnitudes of the same kind have a common measure, we say that—'The ratio of one magnitude to another is the number (integral or fractional) which measures the first, if the second is taken for unit.'

We cannot define exactly what the word ratio, in its most general sense, means; but we can compare two ratios, and determine whether one ratio is greater than, equal to, or less than another. This relationship of equality, or inequality, between ratios is the subject of Euclid's Fifth Book.

That two magnitudes may have a ratio they must be of the *same kind*: as far as *plane* geometry is concerned this means that they must be both lines, or both angles, or both areas. This necessary and sufficient condition is expressed by Euclid in the words—'magnitudes are said to have a ratio to each other which can, being multiplied, exceed the one the other.' In other words they must be such that, if either of them is repeated *often enough*, the sum of its repetitions will exceed the other. We cannot therefore have the ratio of a line to an area, for we cannot make a line exceed an area, however often we repeat the line.

The ratio of one magnitude (A) to another of the same kind (B) may be estimated by examining how the multiples of A are distributed among the multiples of B, when both are arranged in order of magnitude, and the series of multiples continued onwards without limit.

This interdistribution of multiples is definite for two given magnitudes A and B, and is different from that for A and C, if C differs from B by any magnitude however small.

Now as the multiplication of a magnitude, being simply its repetition an assigned number of times, is always possible, the preceding mode of estimation must be always possible. This is the mode used by Euclid.

If the ratio of A to B is given, we are not given A and B themselves, but only the answer to this question, for *all* values of m—

Between what multiples of B lies m A?

To put it more fully—If we form the two scales of multiples

A, 2 A, 3 A, &c., continued indefinitely;

B, 2 B, 3 B, &c., also continued indefinitely;

we know the ratio of A to B if we know the multiples n B and $(n + 1)$ B between which m A lies, for any value of m.

If it should happen that any one of the multiples of B (say n B) is exactly equal to m A, then the quantities A and B are said to be *commensurable*. In that case their treatment falls within the province of arithmetic.

$$\text{Since when } m\,A = n\,B,$$

$$A = \frac{n}{m}\,B,$$

we say—'That relation in virtue of which A is a fraction of B—a fraction being defined as the ratio of two numbers—is called the ratio of A to B, when they are *commensurable*.'

But if it should *not* happen that there are any two terms in the scales, which are equal, so that we only know that m A lies between n B and $(n + 1)$ B, then the quantities A and B are said to be *incommensurable*; and the arithmetical mode of treatment fails entirely.

$$\text{Since when } m\,A \text{ lies between } n\,B \text{ and } (n + 1)\,B,$$

$$A \text{ lies between } \frac{n}{m}\,B \text{ and } \left(\frac{n}{m} + \frac{1}{m}\right) B,$$

and that, as we make m larger, we diminish $1/m$, we say—'That relation in virtue of which A can be expressed as lying between two fractions of B, which fractions can be brought as near together as we please, is called the ratio of A to B, when they are *incommensurable*.'

In the case of commensurables when two ratios are equal to the same fraction, the four magnitudes constituting them are said to be *proportionals*.

$$\text{Thus if } \frac{A}{B} = \frac{n}{m} = \frac{C}{D},$$

then A has to B a ratio which is equal to the ratio of C to D; and A, B, C, D are called *arithmetic proportionals*.

But in the case of incommensurables the above test fails, because there is no common fraction n/m to which the ratios can be equated; and we have to examine how the two pairs of scales of multiples of the magnitudes are connected.

The test given by Euclid is this—

Def. Let there be four magnitudes, of which the first and second are of the same kind, and the third and fourth are of the same kind: let any equimultiples be taken of the first and third; and let any equimultiples be taken of the second and fourth. Then, if it is found that *always* the multiple of the first is greater than, equal to, or less than the multiple of the second, according as the multiple of the third is greater than, equal to, or less than the multiple of the fourth—under those conditions, *the first is said to have* **the same ratio** *to the second that the third has to the fourth*; and the four magnitudes are called **proportionals**.

And, conversely, if four magnitudes are proportionals, and any equimultiples are taken of the first and third, and also any equimultiples of the second and fourth; then the multiple of the first must be greater than, equal to, or less than the multiple of the second according as the multiple of the third is greater than, equal to, or less than the multiple of the fourth.

The foregoing definition has been otherwise expressed thus—'Quantity-ratios are equal if *every* fraction is either equal to both, greater than both, or less than both—a fraction being defined to be the ratio of two numbers.' (Clifford.)

Which may again be abbreviated into this—Quantity-ratios are equal when no fraction lies between them.

Euclid's definition, expressed in the notation already indicated, is the same as this—

The ratio of A to B is equal to that of P to Q

when $\mathbf{m}$ A $>$, $=$, or $<$ $\mathbf{n}$ B according as

$\mathbf{m}$ P $>$, $=$, or $<$ $\mathbf{n}$ Q,

whatever numbers $\mathbf{m}$ and $\mathbf{n}$ may be.

It is an immediate consequence that the ratio of A to B is equal to that of P to Q, when—

if $\mathbf{m}$ A is between $\mathbf{n}$ B and $(\mathbf{n} + 1)$ B, or is equal to $\mathbf{n}$ B,

then also $\mathbf{m}$ P is between $\mathbf{n}$ Q and $(\mathbf{n} + 1)$ Q, or is equal to $\mathbf{n}$ Q;

where $\mathbf{m}$ is any number whatever, and $\mathbf{n}$ is determined by the hypothesis.

Or again the definition may be expressed thus—

The ratio of A to B is equal to the ratio of P to Q when the multiples of A are distributed among the multiples of B in the same manner as the multiples of P are among the multiples of Q.

Note that A and B must be magnitudes of the same kind—areas for instance—also P and Q must be of the same kind, but not necessarily the same as the former two—lines for instance.

Note also that to say

$$mA > nB \quad \text{when, and only when} \quad mC > nD,$$

implies also that

$$mC > nD \quad \text{when, and only when} \quad mA > nB.$$

And similarly for the other inequality.

The proportionality of four magnitudes A, B, C, D will be indicated by the following notation —

$$A : B = C : D,$$

which is to be read thus—'*the ratio of* A *to* B *is equal to (or, is the same as) the ratio of* C *to* D.'

Similarly $A : B > C : D$ is to be read—'*the ratio of* A *to* B *is greater than the ratio of* C *to* D.'

And $A : B < C : D$ is to be read—'*the ratio of* A *to* B *is less than the ratio of* C *to* D.'

Def. When two magnitudes have a ratio to each other, each of them is called a *term* of the ratio.

Def. If $A : B = C : D$, then the terms A and C are called the *antecedents* of the ratios; and the terms B and D are called the *consequents* of the ratios: the terms A and D are called the *extremes* of the proportion; and the terms B and C are called the *means* of the proportion.

Def. The antecedents are said to be *homologous*, because they occupy the same relative position in the ratios; so also the consequents are said to be *homologous*.

From the definition of the equality of ratios it follows that—

1°, equal magnitudes have the same ratio to the same magnitude: for the scales of multiples of the terms of the first ratio are identical with those of the terms of the second ratio.

2°, if two ratios are equal, as the antecedent of the first is greater than, equal to, or less than its consequent, so is the antecedent of the other greater than, equal to, or less than its consequent: for this is contained in the definition, if the multiples taken are unit multiples, that is are the magnitudes themselves.

Def. When of any one set of the multiples of four magnitudes, taken as in the preceding definition, the multiple of the first is greater

than that of the second, but the multiple of the third is not greater than that of the fourth; then the first is said to have to the second a *greater ratio* than the third has to the fourth; and the third is said to have to the fourth a *less ratio* than the first has to the second.

Or, in other words, the ratio of A to B is greater than that of P to Q, when whole numbers m and n can be found, such that, while m A is greater than n B, m P is not greater than n Q; or that while m A is equal to n B, m P is less than n Q.

From this definition it follows that—

$$1^\circ, \quad A + M : B > A : B,$$

$$\text{and } B : A + M < B : A.$$

For let M be multiplied until it exceeds B,

$$\text{suppose} \quad mM = B + K,$$

$$\therefore \quad m(A + M) = mA + B + K.$$

Let mA lie between nB and $(n + 1)B$,

$$\therefore \quad m(A + M) \text{ lies between } nB + B + K \text{ and } (n + 1)B + B + K,$$

$$\text{and } \therefore \text{ is beyond } (n + 1)B;$$

$$\text{i.e. when } m(A + M) > (n + 1)B,$$

$$mA < (n + 1)B;$$

$$\therefore \quad A + M : B > A : B,$$

$$\therefore \quad \text{also } B : A + M < B : A.$$

2°, magnitudes which have the same ratio to the same magnitude are equal. Let A, B, C be three magnitudes of the same kind, such that

$$A : C = B : C.$$

Then, by 1°, if $A > B$, it would follow that

$$A : C > B : C,$$

which is not true.

Also, if $A < B$, it would follow that

$$A : C < B : C,$$

which is not true.

It remains $\therefore$ that $A = B$.

Def. The ratio $A : B$ is termed a *ratio of equality*, *of greater inequality*, or *of less inequality*, according as $A =, >$, or $< B$.

Concerning commensurables and incommensurables it is to be noticed that—

1°, *in*commensurability is the *rule,* and commensurability the *exception*;

2°, though we cannot find an arithmetic fraction which is equal to the ratio of two incommensurable magnitudes, yet we can find a fraction which approximates to that ratio as nearly as we please; and hence that, as we may, by any alteration however minute, convert the latter kind of magnitudes into the former, so to any results which we can prove for commensurables, we may expect a series of collateral and similar results for incommensurables.

Now if the letters all denote commensurable magnitudes, the following propositions admit of easy proof—see Algebra, chapter on Proportion—

1. If $\left.\begin{array}{l}A : B = X : Y,\\ C : D = X : Y,\end{array}\right\}$ and
then $A : B = C : D$.

2. $mA : mB = A : B$.

3. If $A : B = C : D$,
then $B : A = D : C$. *invertendo.*

4. If $A : B = C : D$,
and the *four* magnitudes are of the *same kind,*
then $A : C = B : D$. *alternando.*

5. If $A : B = C : D$,
then $A + B : B = C + D : D$ *componendo.*

6. If $A : B = C : D$,
then $A \sim B : B = C \sim D : D$. *dividendo.*

Cor. to 5 and 6. $A + B : A \sim B = C + D : C \sim D$.

7. If $A : B = C : D = E : F$,
then $A : B = A + C + E : B + D + F$. *addendo.*

8. If $\left.\begin{array}{l}A : B = X : Y,\\ B : C = Y : Z.\end{array}\right\}$ and
Or if $\left.\begin{array}{l}A : B = Y : Z,\\ B : C = X : Y,\end{array}\right\}$ and
then, in both cases,

$$A : C = X : Z. \qquad \textit{ex æquali.}$$

Note—The Latin word, written after each proposition, is the name by which it is usually quoted.

We now proceed to show that all these propositions can be deduced from Euclid's definition, for the cases of all quantities, whether commensurable or incommensurable. And as we shall find that all the propositions of the arithmetic theory of proportion flow from this definition, the appropriateness of the definition will appear.

THEOREM 1—*Ratios that are equal to the same ratio are equal to each other.*

Let A, B, C, D, X, Y be magnitudes such that

$$A : B = X : Y,$$

and $$C : D = X : Y.$$

Take m and n any numbers whatever.

Then	$mA > nB$	when, and only when	$mX > nY$,
and	$mC > nD$	„ „	$mX > nY$;
$\therefore$	$mA > nB$	„ „	$mC > nD$.
Similarly	$mA < nB$	„ „	$mC < nD$,
and	$mA = nB$	„ „	$mC = nD$.

But these are the conditions that

$$A : B = C : D,$$

which is $\therefore$ true.

THEOREM 2—*The ratio of equimultiples of two magnitudes is equal to that of the magnitudes themselves.*

Let A and B be two magnitudes of the same kind; and let m, n, p, q be any numbers whatever.

Then

$pA >, =,$ or $< qB$, according as $m \cdot pA >, =,$ or $< m \cdot qB$;

i. e.

$pA >, =,$ or $< qB$, according as $p \cdot mA, >, =,$ or $< q \cdot mB$.

But this is the criterion that

$$A : B = mA : mB,$$

which is $\therefore$ true.

Cor. 1. It is an immediate consequence of this

that if $$A : B = C : D,$$

then $$mA : mB = nC : nD.$$

Cor. 2. It can also be shown, as in the theorem,

that $$mA : nB = mC : nD.$$

Def. The ratios $A : B$ and $B : A$ are called *reciprocal ratios.*

Note—If reciprocal ratios are equal the terms of either are equal.

THEOREM 3—*If two ratios are equal their reciprocal ratios are equal.* (*Invertendo.*)

Let A, B, C, D be the terms of two ratios such that

$$A : B = C : D.$$

Take m and n any numbers whatever.

Then $mA > nB$ when, and only when $mC > nD$,

$\therefore$ $nB < mA$ „ „ $nD < mC$.

Similarly $nB > mA$ „ „ $nD > mC$.

And $nB = mA$ „ „ $nD = mC$.

But these are the conditions that

$$B : A = D : C,$$

which is $\therefore$ true.

THEOREM 4—*If four magnitudes of the same kind are proportionals, then, alternately, as the first antecedent is to the second antecedent, so is the first consequent to the second consequent.* (*Alternando.*)

Let A, B, C, D be four magnitudes of the same kind, such that

$$A : B = C : D.$$

Lemma 1. $A : C >$ or $< B : C$ as $A >$ or $< B$,

and $C : A <$ or $> C : B$ as $A >$ or $< B$.

For if $A > B$, we can by multiplying the difference between A and B often enough, make it greater than the finite magnitude C.

i.e. m can be found so that mB is less than mA by a quantity greater than C,

and $\therefore$ if mA lies between nC and $(n + 1)C$, or $= nC$,

mB will be less than nC.

But this is the criterion that

$$A : C > B : C.$$

Next, if $A < B$, then $B > A$,

$$\text{and} \quad B : C > A : C,$$

$$\text{i.e.} \quad A : C < B : C.$$

Again, if $A > B$, then, as before,

$$n C > m B, \text{ but } n C \ngtr m A;$$

$$\therefore \quad C : B > C : A,$$

$$\text{or} \quad C : A < C : B.$$

If $A < B$, then $B > A$, and

$$C : B < C : A,$$

$$\therefore \quad C : A > C : B.$$

Lemma 2. $A >$ or $< C$ as $B >$ or $< D$.

For, by *Lemma* 1, if $A > C$,

$$A : B > C : B,$$

$$\therefore \quad C : D > C : B,$$

$$\therefore \quad B > D.$$

Similarly if $A < C$, $B < D$.

Now by *Theorem* 2. *Cor.* 1,

$$m A : m B = n C : n D.$$

And, by *Lemma* 2, $m A >, =, \text{ or } < n C$,

according as $m B >, =, \text{ or } < n D$.

But this is the criterion that

$$A : C = B : D,$$

which is $\therefore$ true.

THEOREM 5—*If two ratios are equal, the sum of the antecedent and consequent of the first has to its consequent the same ratio as the sum of the antecedent and consequent of the second has to its consequent.* (*Componendo.*)

Let A, B, C, D be the terms of two ratios such that

$$A : B = C : D.$$

1°, let the magnitudes be incommensurable.

Take m any number whatever.

Let mA lie between nB and $(n+1)B$,

so that

$mA + mB$ lies between $mB + nB$ and $mB + (n+1)B$,

i.e. so that

$m(A+B)$ lies between $(m+n)B$ and $(m+n+1)B$. (1)

Then, from the condition of proportionality, we must have, corresponding to the preceding,

mC lying between nD and $(n+1)D$,

or

$m(C+D)$ lying between $(m+n)D$ and $(m+n+1)D$. (2)

But the simultaneous truth of (1) and (2) for all values of m, is the criterion that

$$A + B : B = C + D : D,$$

which is $\therefore$ true for incommensurables.

2°, for commensurables, if $mA = nB$, so also $mC = nD$,

and $\therefore$

$m(A+B) = (m+n)B$ when $m(C+D) = (m+n)D$,

$\therefore$ theorem is true for commensurables also.

THEOREM 6—*If two ratios are equal, the difference between the antecedent and the consequent of the first has to its consequent the same ratio as the difference between the antecedent and consequent of the second has to its consequent.* (*Dividendo.*)

Proof precisely similar to that given for *Componendo*—recollecting to subtract the multiple of the lesser of the two A and B.

Cor. to *componendo* and *dividendo.*

If $\qquad A : B = C : D,$

then $\quad A + B : A \sim B = C + D : C \sim D.$

THEOREM 7—*If any number of ratios are equal, all the magnitudes being of the same kind, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.* (*Addendo.*)

Let A, B, C, D, E, F be the terms of three ratios, such that

$$A : B = C : D = E : F,$$

where the magnitudes are all of the same kind.

Then, if m and n are any numbers whatever,

$$\begin{aligned} & \mathbf{m}A >, =, \text{or} < \mathbf{n}B, \\ \text{accordg. as } & \mathbf{m}C >, =, \text{or} < \mathbf{n}D, \\ \text{also accordg. as } & \mathbf{m}E >, =, \text{or} < \mathbf{n}F; \\ \therefore \text{ also accordg. as } & \mathbf{m}(A + C + E) >, =, \text{or} < \mathbf{n}(B + D + F). \end{aligned}$$

But this is the criterion that

$$A : B = A + C + E : B + D + F,$$

which is ∴ true.

And process can obviously be extended to any number of ratios.

THEOREM 8—*If there is a set of magnitudes all of the same kind, and also another set all of the same kind; such that the first is to the second of the first set as the first to the second of the other set; and the second to the third of the first set as the second to the third of the other; and so on to the last magnitude; then the first is to the last of the first set as the first to the last of the other set.* (*Ex æquali.*)

1°, let the sets consist of three magnitudes each, viz.: A, B, C and X, Y, Z,

$$\left.\begin{aligned} \text{so that } & A : B = X : Y, \\ \text{and } & B : C = Y : Z. \end{aligned}\right\}$$

Lemma. $A >$ or $< C$ according as $X >$ or $< Z$.

For if $A > C$ then $A : B > C : B$, (*Theor.* 4. *Lem.* 1.)

$$\therefore \text{ also } > Z : Y.$$

$$\therefore \quad X : Y > Z : Y,$$

$$\therefore \quad X > Z.$$

Similarly if $A < C$ then $X < Z$.

Now $mA : mB = mX : mY$, (*Theor.* 2. *Cor.* 1.)

and $mB : nC = mY : nZ$, (*Theor.* 2. *Cor.* 2.)

where m and n are any numbers whatever.

$\therefore$, by the *Lemma*, $mA >, =,$ or $< nC$,

accordg. as $mX >, =,$ or $< nZ$.

But this is the criterion that

$$A : C = X : Z,$$

which is $\therefore$ true.

2°, let there be also given, in addition to 1°, that

$$C : D = Z : V.$$

Then combining result of 1° with this, we have

$$A : D = X : V.$$

And, similarly, theorem may be extended to any number of ratios.

$$\textit{Cor.} \text{ If } \left.\begin{aligned} A : B &= Y : Z, \\ \text{and } B : C &= X : Y, \end{aligned}\right\}$$

$$\text{then also } A : C = X : Z.$$

For let Q be a fourth proportional to Y, Z, X, so that

$$Y : Z = X : Q.$$

$$\therefore, \textit{alternando}, \quad Y : X = Z : Q.$$

$$\therefore, \textit{invertendo}, \quad X : Y = Q : Z.$$

$$\therefore \quad A : B = X : Q,$$

$$\text{and } B : C = Q : Z,$$

$$\therefore \quad A : C = X : Z.$$

Def. When any number of magnitudes A, B, C, D, E, &c., are so related that A : B = B : C = C : D = D : E = &c., then the magnitudes are said to be in *continued proportion*; B is said to be a *mean proportional* between A and C; B and C are said to be *two mean proportionals* between A and D; B, C, D are said to be *three mean proportionals* between A and E; and so on.

Also A is said to have to C the *duplicate ratio* of that which A has to B; A is said to have to D the *triplicate ratio* of that which A has to B; and so on.

C is called the *third proportional* to A and B.

THEOREM 9—*If two ratios are equal, their duplicates are also equal; and conversely.*

Let A, B, C, D, X, Y be magnitudes such that

A : B = B : X,

and C : D = D : Y.

1°, if A : B = C : D,

then B : X = D : Y.

∴ *ex æquali* A : X = C : Y.

i.e. the duplicate of A : B = the duplicate of C : D.

2°, if A : X = C : Y,

assume that A : B = C : P = P : Q.

∴ C : Q = dupl. of C : P = dupl. of A : B = A : X = C : Y.

∴ Q = Y.

∴ C : P = P : Y.

But D : C = Y : D.

∴ *ex æquali* D : P = P : D.

∴ P = D.

∴ A : B = C : D.

Note—From Theorem 8 and its Corollary, it follows that, of a set of ratios of magnitudes of the same kind, the ratio

first antecedent : last consequent,

is independent of the other antecedents and consequents, or their order; *provided only* that each magnitude which occurs anywhere as antecedent also occurs somewhere else as consequent.

Hence, *under such conditions*, it produces the same effect to alter a magnitude successively in any number of ratios, as to alter it at once in the ratio first antecedent to last consequent. Reference to this very important principle will be facilitated by the following definition.

Def. When there are any number of magnitudes of the same kind, the first is said to have to the last **a ratio which is compounded** of the ratio of the first to the second, of the ratio of the second to the third, and so on up to the ratio of the last but one to the last.

Thus if A, B, C, &c., X, Y, Z represent the magnitudes, then the ratio A : Z is said to be compounded of the ratios

$$A : B, \quad B : C, \quad \&c. \quad X : Y, \quad Y : Z.$$

We shall denote this, for brevity, by the notation

$$A : Z = (A : B)\ (B : C)\ (C : D)\ \&c.\ (X : Y)\ (Y : Z).$$

Furthermore, if $A : B = \alpha : \beta$,

and $C : D = \beta : \gamma$,

then, in accordance with the foregoing definition and notation, (A : B) (C : D) will be properly interpreted to mean *the ratio which is compounded of ratios, of the form* $\alpha : \beta$ *and* $\beta : \gamma$, *that are the same with the ratios* A : B *and* C : D.

Similarly for the rest of the magnitudes.

This may be briefly expressed as follows—

$$(A : B)\ (C : D)\ (E : F)\ \&c.\ (W : X)\ (Y : Z)$$

= the ratio which is compounded of the ratios

$$A : B, \quad C : D, \quad E : F, \quad \&c., \quad W : X, \quad Y : Z.$$

Note (1)—By this definition of *compound ratio*, we see that *Theorem* 8 and its *Cor.* can be both included in this brief enunciation—*Two ratios which are compounded of two sets of equal ratios are themselves equal.*

Note (2)—The ratio which is compounded of reciprocal ratios is a ratio of equality, that is unity.

For $(A : B)\ (B : A) = A : A$.

Note (3)—That duplicate ratio is a ratio compounded of two equal ratios, or a ratio compounded with itself, appears at once thus—

For *any* three magnitudes A, B, C, of the same kind,

$$A : C = (A : B)\,(B : C)$$

$\therefore$ if $A : B = B : C$

then $A : C = (A : B)\,(A : B)$

But then also $A : C$ is, by def., the dupl. of $A : B$

$\therefore$ the dupl. of $A : B = (A : B)\,(A : B)$

So also the triplicate of $A : B = (A : B)\,(A : B)\,(A : B)$

Note (4)—If the letters all denote numbers, integral or fractional, then

the duplicate ratio of m to n is $m^2 : n^2$,

„ triplicate „ „ $m^3 : n^3$;

and the ratio compounded of any number of ratios as a to b, c to d, e to f, &c. is product of all antecedents : product of all consequents,

i. e. is ace &c. : bdf &c.

But it is to be carefully recollected that to use the word 'product' is always tacitly to assume that the quantities to which it is applied can be represented by numbers.

THEOREM 10—*If* A, B, C, D, X, Y *are magnitudes such that*

$$A : X = C : Y,$$

and $B : X = D : Y$;

then $A + B : X = C + D : Y$.

For since $A : X = C : Y$,

and, *invertendo*, $X : B = Y : D$;

$\therefore$, *ex æquali*, $A : B = C : D$.

$\therefore$, *componendo*, $A + B : B = C + D : D$.

But $B : X = D : Y$.

$\therefore$, *ex æquali*, $A + B : X = C + D : Y$.

Note—Although any treatment of the ratios of Geometric Magnitudes (which vary *continuously*) based on their representation by Arithmetic Numbers (which vary *discontinuously*) is entirely fallacious—still (as in the Notes on the two Theorems proved in the Addenda following) by making the unit of measurement *small enough*, we can find numbers whose ratio will represent any proposed geometric ratio, *to any assigned degree of accuracy*.

ADDENDA TO BOOK V.

When four magnitudes are of the *same kind*, the principle *alternando* can be used; and, in that case, *ex æquali* and *componendo* can be much more easily proved than when the second pair of magnitudes are of a different kind from the first pair.

$$\text{Thus let } \mathsf{A : B = X : Y},$$
$$\text{and } \mathsf{B : C = Y : Z};$$

where *all* the magnitudes are of the *same kind*.

$$\text{Then } \textit{alternando} \quad \mathsf{A : X = B : Y},$$
$$= \mathsf{C : Z}, \text{ similarly.}$$
$$\therefore \quad \mathsf{A : C = X : Z};$$

which is *ex æquali*.

Again, let A, B, C, D be four magnitudes of the *same kind*, such that

$$\mathsf{A : B = C : D},$$
$$\therefore \ \textit{alternando} \quad \mathsf{A : C = B : D}.$$
$$\therefore \ \textit{addendo} \quad \mathsf{A + B : C + D = A : C},$$
$$= \mathsf{B : D},$$
$$\therefore \ \textit{alternando} \quad \mathsf{A + B : B = C + D : D};$$

which is *componendo*.

It has been stated, in Book v, that pairs of magnitudes of the same kind exist, which are incapable of being measured by any (the same) unit: or, in other words, that there is no unit of measurement which is contained in each of such a pair an exact number of times.

We proceed to demonstrate this in two particular cases.

LEMMA—*If a magnitude* X *measures each of two magnitudes* A *and* B, *then* X *also measures the difference of* A *and* B.

For let A contain X m times,
and B ,, n ,,
so that $\mathsf{A} = \mathrm{m} \,.\, \mathsf{X}$
and $\mathsf{B} = \mathrm{n} \,.\, \mathsf{X}$,

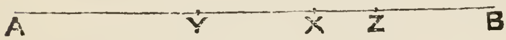
$$\therefore \quad \mathsf{A} \sim \mathsf{B} = (\mathrm{m} \sim \mathrm{n})\,\mathsf{X}.$$

$\therefore$ A ~ B contains X m ~ n times:

i. e. X measures A ~ B.

THEOREM (1)—*The segments of a line divided in medial section* (ii. 11) *are incommensurable.*

It was shown Cor. to ii. 11 (p. 99) that if a line AB is divided in medial section in X (AX being the greater segt.) and if Y is taken in AX, so that AY = BX, then AX is divided in medial section in Y, and also BY in X.



Simrly. if Z is taken in BX, so that BZ = XY, then BX and YZ are each divided in medial section.

Also the following facts are evident—

$$\begin{cases} AX > BX, \text{ but } < 2\,BX. \\ YZ < \frac{1}{2}\,AB. \\ AX - BX = YX. \\ BX - YX = ZX. \end{cases}$$

Whence, 1°, BX does not measure AX;

and, 2°, if there is any finite line M which measures AX and BX, then (by the *Lemma*) M measures YX and ZX, the parts of a line divided in medial section, and which $< \frac{1}{2}$ AB.

i. e. M measures ZX, which $< \frac{1}{4}$ AB.

And, after p repetitions of this process, we should get that M measures a line which $< \frac{1}{4^p} \cdot AB$; i. e. that a finite line measures a line which can be made as small as we please. But this is absurd.

∴ M has no existence:

i. e. AX, BX are incommensurable.

NOTE—$$\frac{AX}{BX} = 1 + \frac{BZ}{BX} = 1 + \cfrac{1}{\cfrac{BX}{BZ}} = 1 + \cfrac{1}{1 + \cfrac{XZ}{BZ}} = \&c.$$

$$= 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\;\&c.$$

And the successive convergents to this continued fraction are

$$\tfrac{1}{1},\ \tfrac{2}{1},\ \tfrac{3}{2},\ \tfrac{5}{3},\ \tfrac{8}{5},\ \tfrac{13}{8},\ \&c.$$

So that the ratio (greater segt. : lesser segt.) is more and more nearly approximated to by taking the ratio of each successive term of the following series to the one that precedes it—

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, &c.,

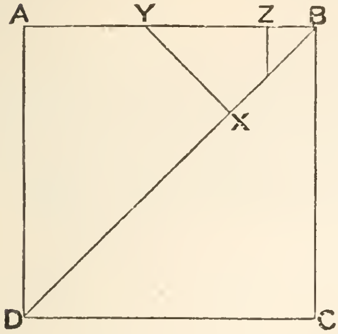
where it will be found that each term is the sum of the two terms preceding.

Thus, if 89 represents the whole line,

then $89 \times 34 = 3026$,

and $55^2 = 3025$.

THEOREM (2)—*A side and a diagonal of a square are incommensurable.*



Let ABCD be a sq.

Take X in diag. DB so that

$$DX = AB.$$

Draw XY, $\perp$ to DB, to meet AB in Y.

Take Z in YB so that

$$YZ = BX.$$

Then the following geometric facts are either obvious, or easily proved—

$$\left\{\begin{array}{l} BX = XY = YA. \\ BY \text{ is diag. of sq. on } BX. \\ DB > AB, \text{ but } < 2\,AB. \\ BX < \tfrac{1}{2}\,AB. \\ BD - AB = BX. \\ AB - BX = BY. \end{array}\right.$$

Whence, 1°, AB does not measure BD;

and, 2°, if there is any finite line M which measures AB and BD, then (by the *Lemma*) M measures BX and BY, the side and diag. of a sq. whose side $< \frac{1}{2}$ AB.

Simrly. M measures BZ, which $< \frac{1}{2}$ BX, i.e. which $< \frac{1}{4}$ AB.

And, after p repetitions of the process, we should get that M measures a line which $< \dfrac{1}{2^p} \cdot AB$; i.e. that a finite line measures a line which can be made as small as we please. But this is absurd.

$\therefore$ M has no existence:

i.e. AB, BD are incommensurable.

NOTE—$\dfrac{BD}{BA} = 1 + \dfrac{BX}{BA} = 1 + \dfrac{1}{\dfrac{BA}{AY}} = 1 + \dfrac{1}{2 + \dfrac{BZ}{YZ}} = \&c.$

$$= 1 + \frac{1}{2+}\,\frac{1}{2+}\ \&c.$$

The successive convergents to which are

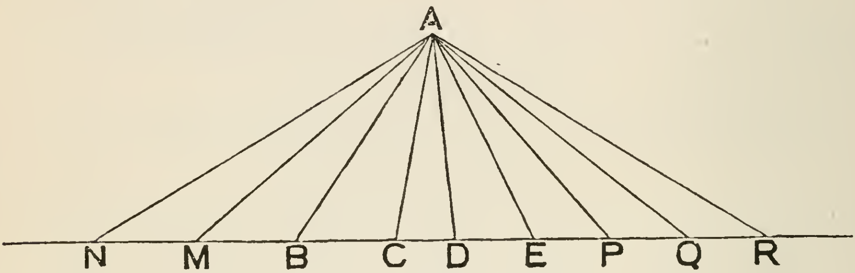
$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \&c.$$

The ratio (diagonal : side) is in fact what is denoted by the symbol $\sqrt{2}$.

BOOK vi.

Proposition 1.

THEOREM—*If two triangles have the same altitude, then the ratio which one triangle has to the other is equal to the ratio which the base of the first has to the base of the second.*



Let ABC, ADE be $\triangle$s which have the same altitude—viz. the $\perp$ from A on their common line of base BCDE.

In the production of CB set off any number of parts,

BM, MN, each of which = BC.

In the production of DE set off any number of parts,

EP, PQ, QR, each of which = DE.

Join A to each of the pts. M, N, P, Q, R.

Then $\because$ CB = BM = MN;

$\therefore \quad \triangle \text{ ACB} = \triangle \text{ ABM} = \triangle \text{ AMN};$

$\therefore$ $\triangle$ ANC and line NC are equimults. of $\triangle$ ABC and base BC.

Similarly

$\triangle$ ARD and line RD are equimults. of $\triangle$ ADE and base DE.

And $\quad \triangle \text{ ANC} >, =, \text{or} < \triangle \text{ ARD},$

according as $\quad \text{NC} >, =, \text{or} < \text{RD}.$

But this is the criterion that

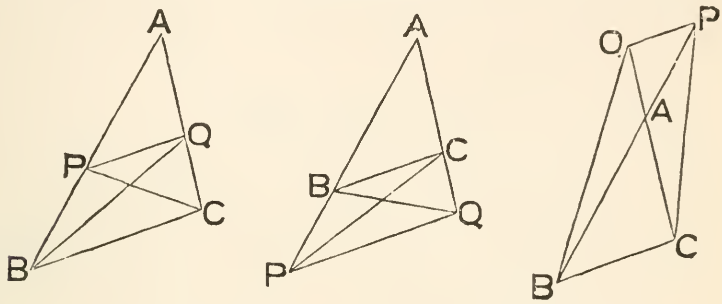
$\triangle \text{ ABC} : \triangle \text{ ADE} = \text{BC} : \text{DE},$

which is $\therefore$ true.

Proposition 2.

THEOREMS—(α) *A straight line parallel to one side of a triangle cuts the other two sides (or these produced) proportionally; so that the segments terminated at the point of concurrence of the latter two sides are homologous:*

(β) *the converse of this is also true.*



(α) Let PQ, $\parallel$ to BC, one of the sides of $\triangle$ ABC, cut sides AB, AC in P, Q respecty. Join BQ, CP.

Then $\triangle$ PQB = $\triangle$ PQC;

$\because$ they are on same base, and between same $\parallel^s$.

$\therefore$ $\triangle$ PQB : $\triangle$ PQA = $\triangle$ PQC : $\triangle$ PQA.

But $\triangle$ PQB : $\triangle$ PQA = BP : PA;

And $\triangle$ PQC : $\triangle$ PQA = CQ : QA.

$\therefore$ BP : PA = CQ : QA.

(β) Next let PQ cut AB, AC so that

BP : PA = CQ : QA.

Then $\triangle$ PQB : $\triangle$ PQA = BP : PA;

and $\triangle$ PQC : $\triangle$ PQA = CQ : QA.

$\therefore$ $\triangle$ PQB : $\triangle$ PQA = $\triangle$ PQC : $\triangle$ PQA.

$\therefore$ $\triangle$ PQB = $\triangle$ PQC.

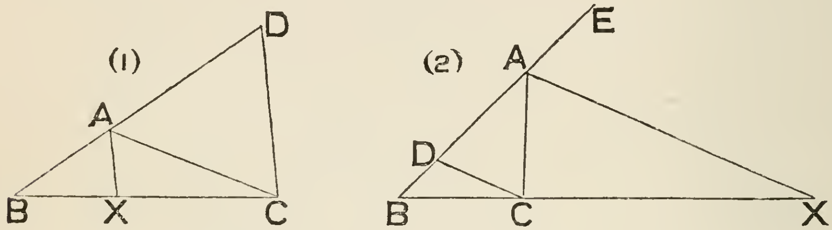
But they are on same base PQ.

$\therefore$ PQ is $\parallel$ to BC.

Proposition 3.

THEOREMS—(α) *If the vertical angle of a triangle is bisected, internally or externally, by a straight line which also cuts the base, then the base is divided internally or externally in the ratio of the sides of the triangle; so that each segment and its conterminous side are homologous:*

(β) *the converse of this is also true.*



(α) Let ABC be a $\triangle$; and let AX bisect

$B\hat{A}C$ in fig. (1),

or $C\hat{A}E$, external to $B\hat{A}C$, in fig. (2).

Draw CD $\parallel$ to AX, meeting BA, or BA produced, in D.

Then $A\hat{D}C = B\hat{A}X$ in fig. (1), or $= E\hat{A}X$ in fig. (2),

$\therefore \quad = C\hat{A}X$ in both figs.

$= A\hat{C}D$.

$\therefore \quad AD = AC$.

And $\because$ AX is $\parallel$ to CD;

$\therefore \quad BA : AD = BX : XC$;

$\therefore \quad BA : AC = BX : XC$.

(β) Next let AX meet BC, fig. (1), or BC produced, fig. (2),

so that $BA : AC = BX : XC$.

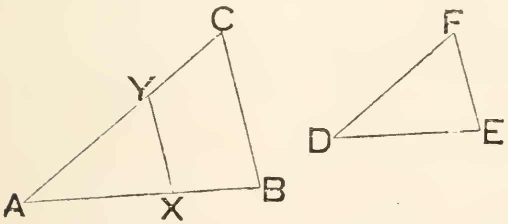
Constructing as before, we have

$BA : AD = BX : XC$.

$$\therefore \quad BA : AC = BA : AD.$$
$$\therefore \quad AC = AD.$$
$$\therefore \quad A\hat{C}D = A\hat{D}C,$$
$$= B\hat{A}X \text{ in fig. (1), or } E\hat{A}X \text{ in fig. (2).}$$
$$\text{But} \quad A\hat{C}D = C\hat{A}X.$$
$$\therefore \quad C\hat{A}X = B\hat{A}X \text{ in fig. (1), or } E\hat{A}X \text{ in fig. (2):}$$
i.e. AX bisects $B\hat{A}C$, or $C\hat{A}E$ external to $B\hat{A}C$.

Proposition 4.

THEOREM—*If two triangles are equiangular to each other, the sides which contain any one of the angles of the one, are proportional to the sides which contain that angle which is equal to it in the other; and those sides which are opposite equal angles are homologous terms in the ratios.*



Let ABC, DEF be $\triangle$s which have $\angle$s at A, B, C and at D, E, F, respectively equal.

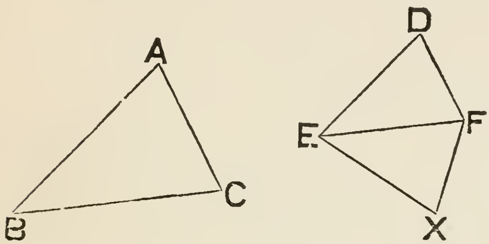
$\because \hat{A} = \hat{D}$, we can place $\triangle\, DEF$ on $\triangle\, ABC$, so that D may be on A, DE on AB, and DF on AC. Then E will take a position X in AB, or AB produced; and F will take a position Y in AC, or AC produced.

$$\text{And} \because \quad A\hat{X}Y = \hat{E} = \hat{B};$$
$$\therefore \quad XY \text{ is } \parallel \text{ to } BC.$$
$$\therefore \quad AB : AC = AX : AY,$$
$$= DE : DF.$$

Similarly for sides about other pairs of equal $\angle$s.

Proposition 5.

THEOREM—*If the sides of two triangles about each of two of their angles are proportional, the triangles are equiangular to each other; and those angles which are opposite to the homologous sides are equal.*



Let ABC, DEF be $\triangle$s such that

$$AB : BC = DE : EF,$$

and $BC : CA = EF : FD;$

$\therefore$ *ex æquali* $AB : CA = DE : FD.$

At pts. E, F, in st. line EF, draw EX, FX so that

$$F\hat{E}X = \hat{B}, \text{ and } E\hat{F}X = \hat{C};$$

$\therefore$ also $\hat{X} = \hat{A}.$

Then, $\triangle$s XEF, ABC being equiang. to each other, gives

$$XE : EF = AB : BC,$$
$$= DE : EF.$$
$$\therefore XE = DE.$$

Similarly $XF = DF.$

And EF is common to $\triangle$s XEF, DEF.

$$\therefore \triangle XEF \equiv \triangle DEF.$$
$$\therefore D\hat{E}F = X\hat{E}F = A\hat{B}C,$$
$$D\hat{F}E = X\hat{F}E = A\hat{C}B,$$

and $\hat{D} = \hat{A}.$

Def. Rectilineal figures which have, 1°, their angles (taken successively) equal, each to each; and, 2°, the sides about the equal angles proportional, in such a manner that the pairs of sides (correspondingly situated with respect to the equal angles) are homologous terms in the ratios, are called **similar.**

Note (1)—The foregoing definition is convertible into the following useful form—*Rectilineal figures are said to be* similar, *when,* 1°, *their angles (taken successively) are equal, each to each; and,* 2°, *the ratio of any side of the one to the side of the other (correspondingly situated with respect to the equal angles) is a constant ratio.*

Note (2)—Of these two necessary and sufficient conditions of similarity, it follows, from Props. 4 and 5, that if a triangle has one it must have the other; so that either is sufficient to ensure the similarity of triangles.

Note (3)—It will be found hereafter [vi. *Addenda* (11), (12), (3)] that if from a fixed point S a variable line SPQ is drawn; then (SP : SQ being constant) if the Locus of P is a line, circle, or rectilineal figure, the corresponding Locus of Q is a line, circle, or similar rectilineal figure—particular cases of the general Theorem—*If from a fixed point* S, *a variable line* SPQ *is drawn, and the ratio of* SP *to* SQ *is constant; then the Loci of* P *and* Q *are similar figures.*

Def. The point S is called a **centre of similarity** of the figures.

Note (4)—The construction of Exercise 77, p. 186, gives a solution of this Problem—*Given two finite lines, find a point which is the common vertex of similar triangles having them for bases.*

Now if A, B, C, D, &c., are a number of points in a diagram; and **a, b, c, d,** &c., the corresponding points on another diagram, representing the same configuration on a different scale—so that ABCD &c., and **abcd** &c., are similar figures—then if the point S is found, at which any corresponding pair of lines AB, **ab**, subtend similar triangles, every pair of joins of corresponding points subtend S in similar triangles. For the join of every pair of points on the one diagram is to the join of the corresponding points on the other diagram in the ratio of the scales of the diagrams. Hence any triangle as ABC is similar to the corresponding triangle **abc**.

$$\therefore\ \widehat{CAS} = \widehat{CAB} + \widehat{SAB} = \widehat{cab} + \widehat{Sab} = \widehat{caS}.$$

$$\text{Also}\quad AC : ac = AB : ab = AS : aS.$$

$$\therefore\ AC : AS = ac : aS.$$

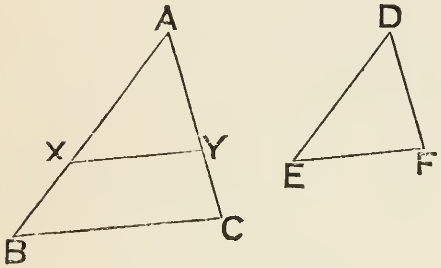
$$\therefore\ \triangle\ CSA \text{ is simr. to } \triangle\ cSa, \text{ by vi. 6.}$$

And similarly for all the points.

This point S is the **centre of similarity** of the two diagrams.

Proposition 6.

THEOREM—*If two triangles have one angle of the one equal to one angle of the other, and the sides about these angles proportional, the triangles are similar; and those angles which are opposite to the homologous sides are equal.*



Let ABC, DEF be $\triangle^s$ such that

$$\left.\begin{aligned} \hat{A} &= \hat{D}, \\ \text{and}\quad AB : AC &= DE : DF. \end{aligned}\right\}$$

Let $\triangle$ DEF be so placed on $\triangle$ ABC that the equal $\wedge^s$ are coincident, and the homologous sides in the same direction.

$\therefore$ E will take a position X on AB, or AB produced;
and F will take a position Y on AC, or AC produced.

Then $AB : AC = DE : DF,$
$= AX : AY.$

$\therefore$ XY is $\parallel$ to BC.

$\therefore$ $A\hat{B}C = A\hat{X}Y = D\hat{E}F,$

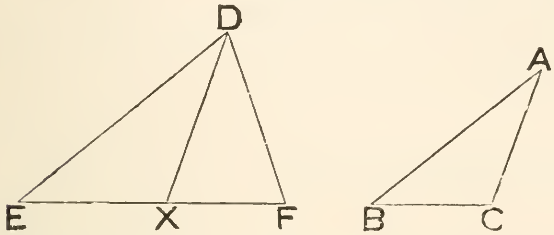
and $A\hat{C}B = A\hat{Y}X = D\hat{F}E.$

$\therefore$ $\triangle^s$ ABC, DEF are equiang. to each other,

and $\therefore$ simr.

Proposition 7.

THEOREM—*If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite the equal angles are homologous, the triangles have their third angles either equal or supplementary; and in the former case the triangles are similar.*



Let ABC, DEF be $\triangle^s$ such that

$$\left.\begin{aligned} \hat{B} &= \hat{E}, \\ \text{and}\quad AB : AC &= DE : DF. \end{aligned}\right\}$$

If $\widehat{BAC} = \widehat{EDF}$, the $\triangle^s$ are equiang. to each other, and $\therefore$ are simr.

If not, assume that $\widehat{EDF} > \widehat{BAC}$.

Draw DX, so that $\widehat{EDX} = \widehat{BAC}$, and X is in EF.

$\therefore\ \triangle$ DEX is equiang. to $\triangle$ ABC.

$$\begin{aligned} \therefore\quad DE : DX &= AB : AC, \\ &= DE : DF. \end{aligned}$$

$$\therefore\quad DX = DF.$$

$$\therefore\quad \widehat{DXF} = \widehat{DFX}.$$

But $\widehat{DXF}$ is suppt. of $\widehat{DXE}$, or of $\widehat{ACB}$.

$\therefore\ \widehat{DFE}$ is suppt. of $\widehat{ACB}$:

i.e. $\hat{C}$ is either equal or supplementary to $\hat{F}$, and in former case $\triangle^s$ are simr.

Note—It is easily seen that the $\triangle^s$ are simr., in the preceding Prop., under any one of the following conditions—

1°, if the $\wedge^s$ given equal are right or obtuse; for then remg. $\wedge^s$ must be both acute,

and $\therefore$ cannot be supplementary.

2°, if the $\wedge^s$ opposite to the other two homologous sides are of same species;

for then they cannot be supplementary.

3°, if the side opposite the given $\wedge$ in each $\triangle$ is not less than the other side which with it form one of the equal ratios;

for then given $\wedge^s$ must be not less than third $\wedge^s$;

$\therefore$ third $\wedge^s$ must both be acute,

and $\therefore$ cannot be supplementary.

Cf. i. *Addenda* (9).

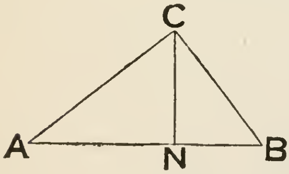
Proposition 8.

THEOREMS—*In a right-angled triangle, if a perpendicular is drawn from the right angle on the hypotenuse—*

(α) *the triangles on each side of it are similar to the whole and to each other:*

(β) *the perpendicular is a mean proportional between the segments of the hypotenuse:*

(γ) *each of the sides is a mean proportional between the hypotenuse and its segment adjacent to that side.*



Let ABC be a $\triangle$, having $\hat{C}$ right.

Draw CN $\perp$ to AB.

Then (α) in $\triangle^s$ ACB, ANC we have

$$\left.\begin{array}{l} A\hat{C}B = A\hat{N}C, \\ \text{and } \hat{A} \text{ common}; \end{array}\right\}$$

∴ $\triangle^s$ are equiang. to each other.

Similarly $\triangle$ ABC is equiang. to $\triangle$ CBN.

∴ the three $\triangle^s$ ABC, ACN, CBN are equiang. to each other,

and ∴ are simr. $\triangle^s$.

(β) from similarity of $\triangle^s$ ANC, CNB, we have

$$AN : CN = CN : BN.$$

(γ) from similarity of $\triangle^s$ ACB, ANC, we have

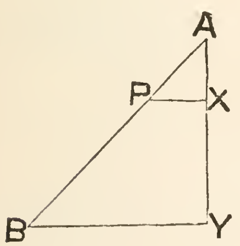
$$AB : AC = AC : AN,$$

and from similarity of $\triangle^s$ ACB, CNB, we have

$$AB : BC = BC : BN.$$

Proposition 9.

PROBLEM—*From a given straight line to cut off any assigned submultiple.*



Let AB be the given st. line.

From A draw any st. line AX, making any $\wedge$ with AB; and produce AX to Y so that AY may be the same mult. of AX that AB is of its assigned sub-mult.

Join BY; and draw XP || to BY to meet AB in P.

Then since PX is || to BY,

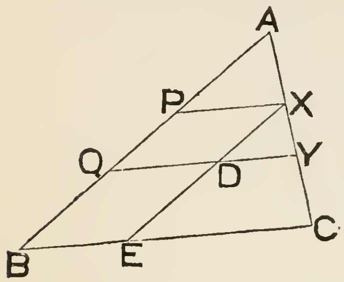
$$AP : AB = AX : AY.$$

∴ AP is the same sub-mult. of AB that AX is of AY;

i.e. AP is the assigned sub-mult. of AB.

Proposition 10.

PROBLEM—*To divide a given straight line into parts proportional to the parts of a given divided straight line.*



Place AB, the given st. line which is to be divided, and AC, the given divided line, so as to form an $\wedge$.

Join BC.

First suppose AC divided into three parts in X, Y.

Draw XP, YQ $\parallel$ to BC and meeting AB in P, Q.

Draw XDE, $\parallel$ to AB, meeting QY in D, and BC in E.

Then figs. PD, QE are $\square^s$.

$\therefore$ XD = PQ, and DE = QB.

Now $\because$ DY is $\parallel$ to EC;

$\therefore$ CY : YX = ED : DX,

= BQ : QP.

And $\because$ PX is $\parallel$ to QY;

$\therefore$ YX : XA = QP : PA.

$\therefore$ also, *ex æquali*, CY : XA = BQ : PA.

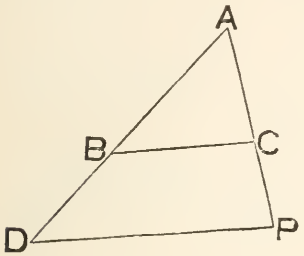
$\therefore$ AB is divided in P and Q into parts respecty. propl. to those into which AC is divided in X and Y.

Next, if CY were divided again in Z, we could, by preceding, divide PB into parts propl. to XY, YZ, ZC.

Similarly the process might be extended to any number of parts.

Proposition 11.

PROBLEM—*To find a third proportional to two given straight lines.*



Place the given lines AB, AC so as to form an $\angle$.

Produce AB to D so that

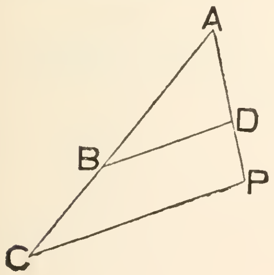
BD = AC.

Join BC; and draw DP $\parallel$ to BC, and meeting AC produced in P.

Then $\because$ BC is $\parallel$ to DP;
$\therefore$ AB : BD = AC : CP;
i. e. AB : AC = AC : CP.
$\therefore$ CP is a third propl. to AB, AC.

Proposition 12.

PROBLEM—*To find a fourth proportional to three given straight lines.*



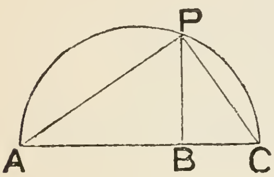
Place the given lines so that the first two of them AB, BC may be in a st. line, and the third AD may make any $\angle$ with the first AB.

Join BD; and draw CP $\parallel$ to BD, and meeting AD produced in P.

Then, since BD is $\parallel$ to CP,
$\therefore$ AB : BC = AD : DP;
i. e. DP is a fourth propl. to AB, BC, AD.

Proposition 13.

PROBLEM—*To find a mean proportional between two given straight lines.*



Place the given lines AB, BC in the same st. line ABC.

On AC as diam. describe a semi-⊙; and let BP, ⊥ to AC, meet this semi-⊙ in P. Join AP, CP.

Then $A\hat{P}C$, being in a semi-⊙, is right.

∴ PB, being drawn from P ⊥ to AC, is a mean propl. between AB, BC.

Def. Two sides, forming an angle of one rectilineal figure, are said to be **reciprocally proportional** to two sides, forming an angle of another rectilineal figure, when a side of first is to a side of second as remaining side of *second* is to remaining side of first.

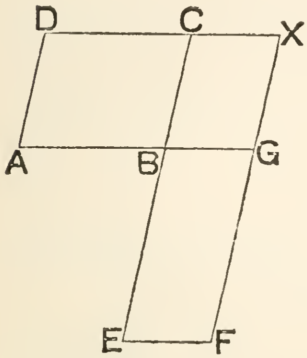
Note—In more general terms, two magnitudes A, B, are said to be **reciprocally proportional** to two other magnitudes X, Y, when—

$$A : X = Y : B.$$

Proposition 14.

THEOREMS—(*a*) *Parallelograms of equal area which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional:*

(β) *and conversely, if two parallelograms have an angle of the one equal to an angle of the other, and the sides about the equal angles reciprocally proportional, the parallelograms have the same area.*



Let ▱s ABCD, BEFG be of equal area, and such that

$$A\hat{B}C = E\hat{B}G.$$

Place them so that AB, BG are in a st. line; and ▱s on opposite sides of ABG. Then EB, BC will also be in a st. line.

Complete ▱ CBGX.

Then AB : BG = ▱ AC : ▱ BX,
= ▱ BF : ▱ BX,
= BE : BC.

∴ (α) is true.

Again, constructing as before, and assuming that

AB : BG = BE : BC;

we have ▱ AC : ▱ BX = AB : BG,
= BE : BC,
= ▱ BF : ▱ BX.

∴ ▱ AC = ▱ BF.

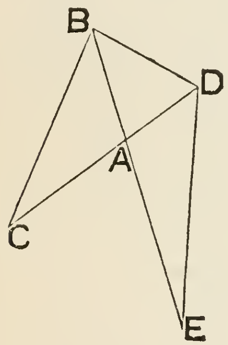
∴ (β) is true.

Note—A *second converse* to (α) can be got by assuming the parallelograms of equal area, and with the sides about a pair of angles reciprocally proportional: then it can be proved [vi. *Addenda* (4)] that these angles are equal.

Proposition 15.

THEOREMS—(α) *Triangles of equal area which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional:*

(β) *and conversely, if two triangles have an angle of the one equal to an angle of the other, and the sides about the equal angles reciprocally proportional, the triangles have the same area.*



Let $\triangle$s ABC, ADE be of equal area, and such that $B\hat{A}C = D\hat{A}E$.

Place them so that BA, AE may be in a st. line, and $\triangle$s on opposite sides of BAE. Then CA, AD will also be in a st. line. Join BD.

$$\text{Then}\quad \begin{aligned} CA : AD &= \triangle\, CAB : \triangle\, ABD, \\ &= \triangle\, DAE : \triangle\, ABD, \\ &= EA : AB. \end{aligned}$$

$\therefore$ (α) is true.

Again, constructing as before, and assuming that

$$CA : AD = EA : AB;$$

$$\text{we have}\quad \begin{aligned} \triangle\, CAB : \triangle\, ADB &= CA : AD, \\ &= EA : AB, \\ &= \triangle\, EAD : \triangle\, ADB. \end{aligned}$$

$$\therefore\quad \triangle\, CAB = \triangle\, EAD;$$

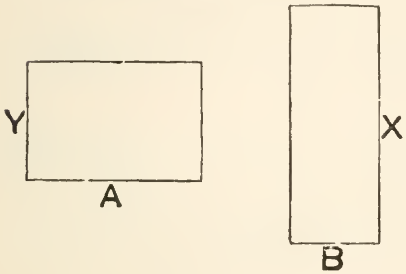
$\therefore$ (β) is true.

Note—As in Prop. 14, a second converse holds.

Proposition 16.

THEOREMS—(α) *If four straight lines are proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means :*

(β) *the converse of this is also true.*



Let A, B, X, Y be four st. lines such that

A : B = X : Y.

Place them so that A and Y are conterminous and $\perp$,

and that B and X are conterminous and $\perp$.

Complete the rects. of which A, Y and B, X are adjacent sides.

Then these rects. are of equal area ;

$\because$ they are equiang. ▱s having their sides about equal $\wedge$ reciprocally propl.

Next, constructing as before, and assuming that

rect. under A, Y = rect. under B, X ;

then, since these rects. are equiang. ▱s of equal area,

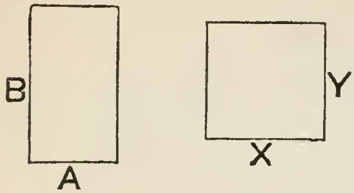
$\therefore$ their sides about their equal $\wedge$s are reciprocally propl.

$\therefore$ A : B = X : Y ;

Proposition 17.

THEOREMS—(α) *If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean :*

(β) *the converse of this is also true.*



Let A, X, B be three st. lines such that

$$A : X = X : B.$$

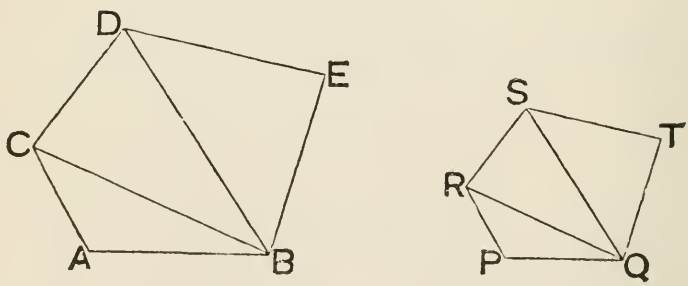
Place them so that A and B are conterminous and ⊥, and draw Y ⊥ to, at an end of, and equal to X.

Complete rects.; and prove exactly as in preceding proposition.

Def. Two similar rectilineal figures are said to be **similarly situated** with respect to a pair of their sides; or two sides of two similar rectilineal figures are said to be **similarly situated** with respect to the figures, when these sides are homologous.

Proposition 18.

PROBLEM—*On a given straight line to describe a rectilineal figure similar to a given rectilineal figure, and so that the given line and an assigned side of the given figure may be similarly situated with respect to the two figures.*



Let AB be the given line; PQ the assigned side of given fig.

1°, let given fig. be △ PQR.

Draw AC, BC, so that

$$B\hat{A}C = Q\hat{P}R, \text{ and } A\hat{B}C = P\hat{Q}R.$$

Then $\triangle$ ABC is equiang., and $\therefore$ simr. to $\triangle$ PQR.

2°, let given fig. be quad. PQSR.

Join RQ dividing the quad. into two $\triangle$s PQR, RQS.

$$\left.\begin{array}{r}\text{On AB make } \triangle \text{ ABC simr. to } \triangle \text{ PQR,} \\ \text{and} \\ \text{on BC make } \triangle \text{ BCD simr. to } \triangle \text{ QRS,}\end{array}\right\} \begin{array}{l}\text{so that } \wedge\text{s at B and C} \\ \text{may be respecty. equal} \\ \text{to } \wedge\text{s at Q and R.}\end{array}$$

$$\begin{aligned} \text{Then} \quad A\hat{C}D &= A\hat{C}B + B\hat{C}D, \\ &= P\hat{R}Q + Q\hat{R}S, \\ &= P\hat{R}S. \end{aligned}$$

$$\text{Similarly} \quad A\hat{B}D = P\hat{Q}S;$$

$\therefore$ fig. ABDC is equiang. to fig. PQSR.

$$\begin{aligned} \text{Also} \quad & AC : CB = PR : RQ, \\ \text{and} \quad & CB : CD = RQ : RS. \\ \therefore \textit{ ex æquali} \quad & AC : CD = PR : RS. \\ \text{Similarly} \quad & AB : BD = PQ : QS. \end{aligned}$$

And sides about $\hat{A}$ and $\hat{P}$ are propl.,

as also are sides about $\hat{D}$ and $\hat{S}$.

$\therefore$ figs. ABDC, PQSR have sides propl. which are about $\wedge$s that are respecty. equal.

$\therefore$ fig. ABDC is simr. to fig. PQSR.

3°, let fig. be pentagon PQTSR.

Then we can describe quad. ABDC simr. to quad. PQSR;

and also $\triangle$ BDE simr. to $\triangle$ QST.

And, as in 2°, it can be shown that

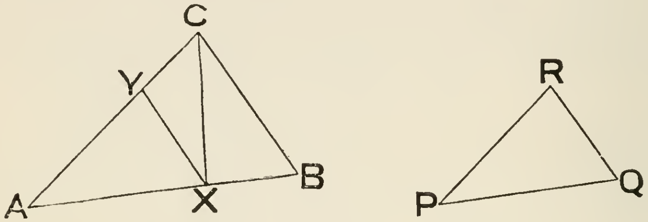
fig. ABEDC is simr. to fig. PQTSR.

And same process can be extended to a fig. of any number of sides.

Also in each case AB and PQ are similarly situated with respect to given figs.

Proposition 19.

THEOREM—*Similar triangles are to each other in the duplicate ratio of their homologous sides.*



Let ABC, PQR be similar $\triangle$s, having $\wedge$s at A, B, C respecty. equal to $\wedge$s at P, Q, R; so that sides AB, PQ are homologous.

Place $\triangle$ PQR on $\triangle$ ABC, so that
P may be on A, PQ on AB, and PR on AC:
this can be done, $\because \hat{P} = \hat{A}$.
Then Q will take a position X, in AB, or AB produced;
R will take a position Y, in AC, or AC produced;
and XY will be $\parallel$ to BC. Join CX.

Let Ω be the 3rd propl. to AB and PQ (or AX) so that
AB : Ω = dupl. ratio of AB : PQ (or AX)

Then $\triangle$ ABC : $\triangle$ AXC = AB : AX = AX : Ω;
and $\triangle$ AXC : $\triangle$ AXY = AC : AY = AB : AX.
$\therefore$ *ex æquali* $\triangle$ ABC : $\triangle$ AXY = AB : Ω,
i.e. $\triangle$ ABC : $\triangle$ PQR = dupl. ratio of AB : PQ.

EXERCISES.

1. If AD is a median of triangle ABC; and DX, DY, the bisectors of angles at D, meet AB, AC respectively in X, Y; then XY is parallel to BC.

2. If the distances of two fixed points from a variable line, are in a fixed ratio, the line must go through a fixed point.

3. ABC is a triangle; a parallel to BC is terminated by the sides AB, AC in X, Y respectively; if BY, CX are joined, and cut in O, then—

1°, $\triangle$ AXO = $\triangle$ AYO; 2°, AO produced bisects BC.

4. Tangents at the ends of a diameter AB of a circle, meet the tangent at any point P in X, Y; if AY, BX cut in Q, then PQ is perpendicular to AB.

5. The rectangle under two lines is a mean proportional between the squares on them.

6. If the bisectors of an opposite pair of angles of a quadrilateral meet on one diagonal, then will the bisectors of the other pair meet on the other diagonal.

7. OA, OB are fixed lines; if any points P, Q are taken in OA, OB respectively; and PR, QS are drawn perpendicular to OB, OA; and RX, SY perpendicular to OA, OB; then XY is parallel to PQ.

8. If two triangles are on opposite sides of the same base, their areas are proportional to the segments of the join of their vertices, made by the base.

9. In a triangle ABC, if M is the mid point of BC; and AD, the bisector of angle A, meets BC in D; then—

$$MB : MD = AB + AC : AB \sim AC.$$

NOTE—*See* i. *Addenda* (26) *for relative positions of* AD, AM.

10. P is any point in the side AB of a triangle ABC; BQ, parallel to CP, meets AC produced in Q; X, Y are points in AB, AC respectively, such that AX is a mean proportional between AB, AP, and AY is a mean proportional between AC, AQ; show that area AXY is equal to area ABC.

NOTE—*Use* vi. 15.

11. A line is drawn from the corner A of a parallelogram ABCD, cutting BD in P, CD in Q, and BC produced in R: show that—

$$PQ : PR = PD^2 : PB^2.$$

NOTE—*Join* BQ; *and use* vi. 1 *and* 19.

12. From the intersection of the diagonals of a cyclic quadrilateral perpendiculars are dropped on a pair of opposite sides: prove that these perpendiculars are in the same ratio as the sides to which they are drawn.

Proposition 20.

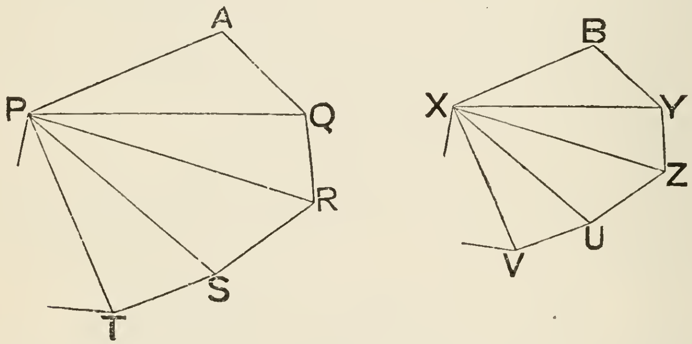
THEOREMS—(α) *Similar polygons may be divided into the same number of similar triangles:*

(β) *the corresponding pairs of triangles in* (α) *have to each other the same ratio that the polygons have:*

(γ) *similar polygons are to each other in the duplicate ratio of their homologous sides:*

(δ) *similar polygons are to each other as the squares on their homologous sides:*

(ε) *similar polygons are to each other as any side of the first is to the third proportional to that side and the homologous side of the second.*



Let PAQRST &c. and XBYZUV &c., be simr. polys. having those $\wedge$s equal which are respecty. in the order of the letters named.

Join P and X with each other corner of its poly.

Then $\triangle$s PAQ, XBY, are simr. ∵

$$\left.\begin{array}{c}\hat{A} = \hat{B}, \\ \text{and sides about these } \wedge^s \text{ are propl.}\end{array}\right\}$$

$$\therefore \quad A\hat{Q}P = B\hat{Y}X.$$

$$\text{But whole } A\hat{Q}R = \text{whole } B\hat{Y}Z;$$

$$\therefore \text{ remg. } P\hat{Q}R = \text{remg. } X\hat{Y}Z.$$

Also since $AQ : QP = BY : YX$,
and $QR : AQ = YZ : BY$;
$\therefore$ *ex æquali* $QR : QP = YZ : YX$.
$\therefore$ $\triangle$s RQP, ZYX are simr.

And simrly. it could be shown that $\triangle$s SRP, UZX are simr.

And the same process could obviously be carried on for each corresponding pair of $\triangle$s round the entire of the polys.

$\therefore$ (α) is true.

Again, $\triangle APQ : \triangle BXY$ in dupl. ratio of PQ to XY;
and $\triangle PQR : \triangle XYZ$ " "
$\therefore$ $\triangle APQ : \triangle BXY = \triangle PQR : \triangle XYZ$.

And similarly it could be shown that each corresponding pair of $\triangle$s in the polys. have the same ratio that each other corresponding pair has.

And one of the antecedents : one of the consequents,
= sum of antecedents : sum of consequents ;
i. e. = poly. PAQ &c. : poly. XBY &c.

$\therefore$ (β) is true.

Again, any $\triangle$, as APQ : corresponding $\triangle BXY$ in dupl. ratio of homologous sides AQ, BY;

$\therefore$ poly. PAQ &c. : poly. XBY &c., in dupl. ratio of $AQ : BY$.

$\therefore$ (γ) is true.

Now if sqs. are described on two homologous sides of the polys., these sqs., being simr. polys., are in dupl. ratio of sides.

$\therefore$ (δ) polys. are as sqs. on their homologous sides.

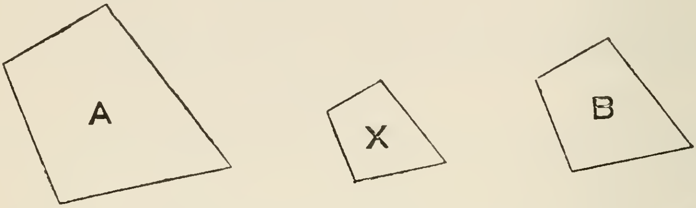
Lastly, if to L, M, any two homologous sides of polys., a third propl. N is taken,

then $L : N$ in dupl. ratio of L to M.
But poly. on L : poly. on M in dupl. ratio of L to M.
$\therefore$ poly. on L : poly. on $M = L : N$.

$\therefore$ (ϵ) is true.

Proposition 21.

THEOREM—*Rectilineal figures which are similar to the same figure are similar to each other.*



Let rectilin. figs. A and B be each simr. to X.

Since A is simr. to X ;

$\therefore$ A is equiang. to X.

Similarly B is equiang. to X.

$\therefore$ A is equiang. to B.

Again, since ratios of pairs of sides about equal $\wedge$s in A and B are each equal to ratio of pair of sides about corresponding equal $\wedge$s in X,

$\therefore$ ratio of sides about an $\wedge$ in A

= ratio of sides about equal $\wedge$ in B.

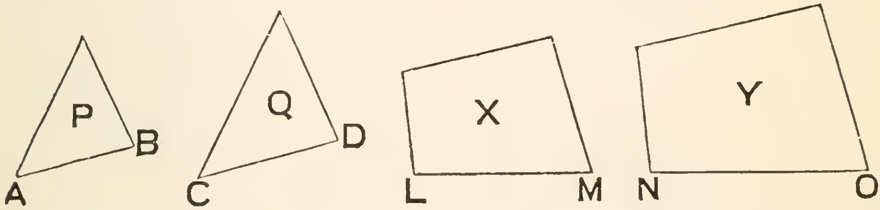
$\therefore$ A is simr. to B.

Note—Before reading the next Prop. the learner should refer to p. 242 for the proof of the Theorem that—*If four magnitudes are proportional, the duplicate ratio of the first to the second is equal to the duplicate ratio of the third to the fourth; and conversely.* That Theorem should be proved as a *Lemma* in connection with the Prop.

Proposition 22.

THEOREMS—(a) *If four straight lines are proportional, and similar rectilineal figures are similarly described on the first and second, and also similar rectilineal figures are similarly described on the third and fourth; then as the figure on the first is to the figure on the second, so is the figure on the third to the figure on the fourth:*

(β) *the converse of this is also true.*



Let AB, CD, LM, NO be four st. lines.

On AB, CD let simr. figs. P, Q be simrly. described,

and on LM, NO let simr. figs. X, Y be simrly. described.

(a) Suppose AB : CD = LM : NO.

Then P : Q = dupl. ratio of AB to CD,

= dupl. ratio of LM to NO,

= X : Y.

(β) Next suppose P : Q = X : Y.

Then dupl. ratio of AB to CD = P : Q,

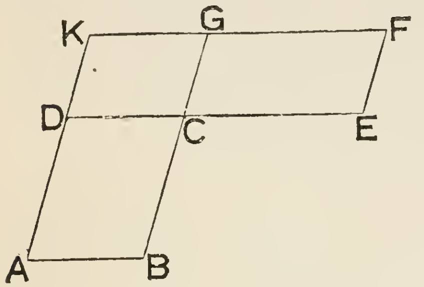
= X : Y,

= dupl. ratio of LM to NO.

∴ AB : CD = LM : NO.

Proposition 23.

THEOREM—*Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.*



Let ABCD, CEFG be equiang. ▱s, in which

$$B\hat{C}D = E\hat{C}G.$$

Place them so that a pair of the lines BC, CG forming the equal ∧s are in a st. line; and the ▱s on opposite sides of that line.

∴ the other pair DC, CE must be in a st. line.

Complete ▱ DCGK.

Take any st. line L; find M the 4th propl. to BC, CG and L; and find N the 4th propl. to DC, CE and M; so that

$$BC : CG = L : M,$$

$$\text{and } DC : CE = M : N.$$

Then L : N is (by def. on p. 243) the ratio compounded of L : M and M : N, that is of BC : CG and DC : CE.

Now ▱ CA : ▱ CK = BC : CG = L : M;

and ▱ CK : ▱ CF = DC : CE = M : N.

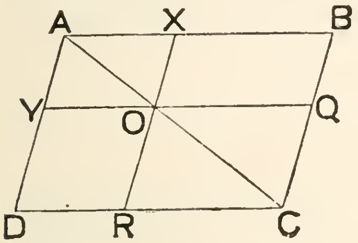
∴ *ex æquali* ▱ CA : ▱ CF = L : N.

Note (1)—The *converse* of Prop. 23 will be found in the *Addenda*.

Note (2)—It is sometimes said that to *compound* ratios is the same as to *multiply* them. This, as a general statement, is quite wrong. The term 'multiply' is an arithmetic term, and though applicable to the ratios of commensurable quantities, has no meaning in relation to the ratios of incommensurables.

Proposition 24.

THEOREM—*In a parallelogram any two parallelograms, which are about either of its diagonals, are similar to the whole and to each other.*



Let ABCD be a ▱; and AXOY, CQOR ▱s about the diag. AOC.

Since $\widehat{AYO} = \widehat{ADC}$, by ∥s YO, DC,

and $\widehat{AOY} = \widehat{ACD}$, „

∴ △ AYO is simr. to △ ADC;

∴ AY : YO = AD : DC.

And since opposite sides of ▱s are equal,

∴ also AY : AX = AD : AB,

and OX : OY = CB : CD,

and AX : XO = AB : BC.

But ∧s of ▱s, about which these propls. respecty. lie, are equal;

∵ ▱s have $\widehat{BAD}$ in common.

∴ ▱ XY is simr. to ▱ BD.

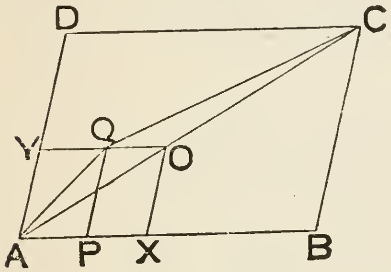
Similarly ▱ QR is simr. to ▱ BD;

∴ also ▱ XY is simr. to ▱ QR.

Note—Prop. 25 will be found immediately after Prop. 26. The order has been changed because Prop. 26, being the *converse* of Prop. 24, should naturally follow it.

Proposition 26.

THEOREM—*If two similar and similarly situated parallelograms have a common angle they are about the same diagonal.*



Let ▱s ABCD, AXOY be simr. and simrly. situated about the common $B\hat{A}D$.

Assume that diag. AQC cuts YO in Q.

Draw QP, ‖ to AY, and meeting AB in P.

Then ▱s BD, PY, being about same diag. AQC, are simr.

$\therefore \quad CB : CD = QP : QY.$

But $\quad CB : CD = OX : OY,$

$= QP : OY.$

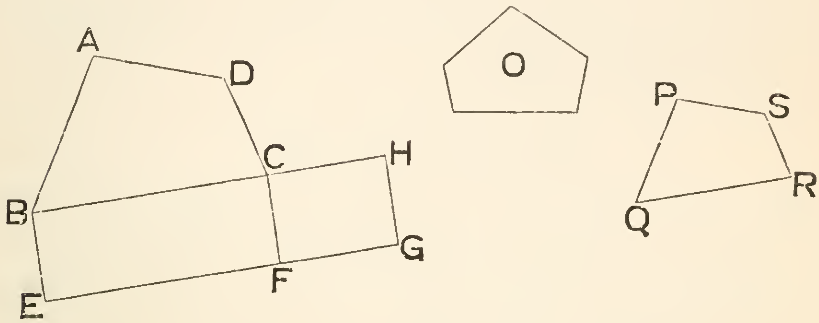
$\therefore \quad OY = QY$, a part of itself.

$\therefore$ the assumption that diag. AC does not go thro. O has led to an absurdity; and $\therefore$ is not true:

i. e. AOC is a common diag. line.

Proposition 25.

PROBLEM—*To describe a rectilineal figure which shall be of given area, and similar to a given rectilineal figure.*



Let ABCD be the given rectilin. fig., O the given area.

On BC describe rect. BCFE, equal to fig. ABCD.

To CF apply the rect. CFGH, equal to fig. O; so that BC, CH are in one line.

Find QR a mean propl. between BC, CH,

so that BC : QR = QR : CH;

and ∴ BC : CH in dupl. ratio of BC to QR.

On QR describe a rectilin. fig. PQRS simr. to ABCD, and so that QR, BC are homologous sides.

Then fig. ABCD : fig. O = rect. BF : rect. CG,

= BC : CH,

= dupl. ratio of BC to QR,

= fig. ABCD : fig. PQRS.

∴ fig. PQRS = fig. O.

i.e. PQRS has been described so as to be similar to ABCD, and of same area as O.

Note—It is the universal custom to omit the three following Props.

Proposition 27.

THEOREM—*Of all the parallelograms applied to the same straight line, and deficient by parallelograms similar and similarly placed to that described upon the half line, that parallelogram is the greatest which is applied to the half line, and is similar to its defect.*

Proposition 28.

PROBLEM—*To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a figure similar to a given parallelogram; but the rectilineal figure must not be greater than the parallelogram applied to half the given line, whose defect is similar to the given parallelogram.*

Proposition 29.

PROBLEM—*To apply to a given straight line a parallelogram equal to a given rectilineal figure, and exceeding by a parallelogram similar to a given one.*

Proposition 30.

PROBLEM—*To divide a given straight line so that the whole line is to the greater segment as the greater segment is to the lesser segment.*

Let AB be the given line.
Divide it in X so that

rect. under AB, BX = sq. on AX.

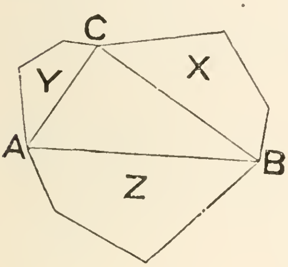
$\therefore$ AB : AX = AX : BX;

i.e. AB is divided in X as reqd.

Def. A straight line divided as in vi. 30 (i.e. as in ii. 11) is said to be divided **in extreme and mean ratio.**

Proposition 31.

THEOREM—*If similar rectilineal figures are similarly described on the three sides of a right-angled triangle, the figure on the hypotenuse is equal to the sum of the figures on the other sides.*



Let ABC be a $\triangle$ having $\hat{C}$ right; and let X, Y, Z denote the areas of simr. rectilin. figs. simrly. described on sides opposite A, B, C respecty.

Since simr. figs. are as sqs. on their homologous sides;

$\therefore$ X : Y = sq. on BC : sq. on AC.

$\therefore$, *componendo*,

X + Y : Y = sq. on BC + sq. on AC : sq. on AC,

= sq. on AB : sq. on AC,

= Z : Y.

$\therefore$ X + Y = Z.

Note—This Prop. can be proved without i. 47—a particular case of itself.

For if CN is $\perp$ to AB, then, by vi. 8, BN : BC = BC : BA;

$\therefore$, by def', BN : BA = dupl' of BC : BA = X : Z, by vi. 20.

Sim'ly AN : AB = dupl' of AC : AB = Y : Z;

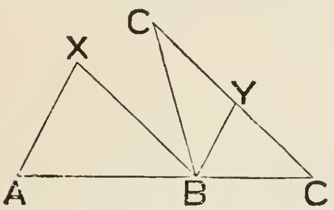
$\therefore$, by v. 10, AN + BN : AB = X + Y : Z.

Proposition 32.

THEOREM—*If two triangles have two sides of the one proportional to two sides of the other, and are so placed at an angle that the homologous sides are parallel, the remaining sides of the triangles are in a straight line.*

This proposition is omitted as quite useless.

Without some further limitation to the given conditions, it is not even necessarily true.



For let AXB, BYC be $\triangle^s$ in which
AX : BX = BY : CY,
and let them be so placed at B that BY is $\parallel$ to AX, and CY to BX.

Then it is clear that either of the positions of $\triangle$ BYC, given in fig., satisfies the stated condns., but that only one of them gives the stated result.

Proposition 33.

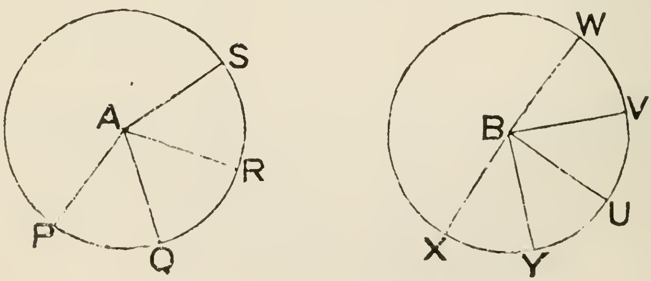
THEOREMS—*In equal circles (or the same circle) the ratio of—*

(α) any two angles at the centres ;

or (β) any two angles at the circumferences ;

or (γ) any two sectors ;

is equal to the ratio of the respective arcs on which they stand.



Let PQ, XY be any arcs of equal ⊙s whose centres are respecty. A and B.

Along circumf. of ⊙, centre A, set off any number of arcs

QR, RS, each of which = PQ.

Along circumf. of ⊙, centre B, set off any number of arcs

YU, UV, VW, each of which = XY.

Join A to each of the pts. R, S; and B to each of pts. U, V, W.

Then (α) $\because$ arc PQ = arc QR = arc RS,

$$\therefore \quad P\hat{A}Q = Q\hat{A}R = R\hat{A}S.$$

$\therefore$ $P\hat{A}S$ and arc PS are equimults. of $P\hat{A}Q$ and arc PQ.

Similarly $X\hat{B}W$ and arc XW are equimults. of $X\hat{B}Y$ and arc XY.

And $P\hat{A}S >, =, \text{or} < X\hat{B}W$,

according as arc PS $>, =,$ or $<$ arc XW.

But this is the criterion that

$$P\hat{A}Q : X\hat{B}Y = \text{arc } PQ : \text{arc } XY,$$

which is $\therefore$ true.

Also (β) an exactly similar process will serve for $\wedge$s which PQ, XY subtend at circumfs.

Or the proportionality of $\wedge$s at the circumfs. to the arcs on which they stand, may be *deduced* from that of the corresponding $\wedge$s at the centres by the consideration that they are equisubmults.—viz. halves—of them.

Lastly (γ) first prove, as a *Lemma*, iii. *Addenda* (8); and then prove the proportionality of sectors APQ, BXY to their arcs, exactly in the same way as (α) is done here.

Note—All the propositions of Euclid's first four, and sixth, Books have been enunciated in the preceding pages. Most of the more familiar intercalations of his various Editors will be found in the *Addenda* to the different Books.

ADDENDA TO BOOK vi.

Corollaries to the Props. in Book vi.

Def. The distance between an opposite pair of sides of a parallelogram (measured by the length of a perpendicular dropped from any point *in* one *on* the other) is called an **altitude** of the parallelogram, with respect to either of those sides considered as base.

Note—Obviously a parallelogram has two altitudes.

vi. 1. (α) Parallelograms of the same altitude are in the same ratio as their bases: this follows at once from the Prop. by considering that the parallelograms are equimultiples (viz. doubles) of triangles of the same altitude; or it can be proved by a precisely similar method to that used in the Prop.

(β) As in the Prop. it could be shown that triangles on equal bases are as their altitudes.

(γ) So also parallelograms on equal bases are as their altitudes.

(δ) Triangles of equal altitude are as their bases.

(ϵ) Parallelograms of equal altitude are as their bases.

(ζ) The Converses of the Prop., and of all the preceding Corollaries, follow easily by *reductio ad absurdum*.

vi. 4. (α) A line drawn across a triangle, parallel to a side, cuts off a similar triangle.

(β) In equiangular triangles the altitudes drawn to homologous sides are proportional to those sides.

(γ) A line drawn from a corner of a triangle (considered as vertex) to meet the base, divides every parallel to the base (terminated by the sides, or sides produced) in the same ratio.

vi. 10. The *external* section of a line, in a given ratio, can be done in exactly the same way.

vi. 13. By this Prop. 3, 7, 15, &c., means may be found between two given lines: for after one mean is found, a mean can be found between it and each of the given lines, thus getting 3 means; then, again, finding means between each successive two of these we get 7 means; and continuing this process we can find $2^n - 1$ means, where n is any positive integer.

Note—The Problem of finding 2 means between two given lines is insoluble by the *un*graduated ruler and compasses alone. On p. 284 will be found one of the many ways of solving it by further mechanical aid.

vi. 16. The principle *alternando* follows at once from this Prop. *in the case of straight lines.* If therefore it can be shown that the ratio of any two magnitudes of the same kind can be represented by the ratio of two straight lines, the principle (for all such magnitudes) is an immediate deduction from this Prop.

Now in the geometry of the Point, Line, and Circle, the only magnitudes that can occur are—

(1) *Lines* (including straight lines, and arcs of circles).

(2) *Angles.*

(3) *Areas* (including rectilineal figures, circles, and sectors of circles).

But to any arc of a circle there is a straight line equivalent in length; and though we cannot (by the use of the ruler and compasses) *find* this line, it clearly has an existence, and might be hypothetically reasoned about. Hence arcs of circles are proportional to straight lines.

Angles again are proportional to arcs of any (the same) circle.

Rectilineal figures can be reduced to rectangles, having a common altitude and therefore proportional to their bases.

Circles can be shown (Euclid xii. 2) to be proportional to the squares on their radii; and therefore come under the conditions of rectilineal figures.

Sectors of the same circle are proportional to their arcs; and sectors of different circles are proportional to the circles they are parts of; and therefore to the squares on their radii.

So that (assuming xii. 2, and the above hypothetical construction) *alternando* follows from vi. 16; and then (as on p. 244) *ex æquali* and *componendo* can be deduced: in this way Book v. might be dispensed with.

Note—It is however to be carefully noted that Euclid does not permit the use of hypothetical constructions; and therefore that to introduce such, is to travel outside the limitations of geometrical reasoning which *he* has laid down—though not necessarily to be illogical.

vi. 20. The perimeters of similar rectilineal figures are proportional to their homologous sides.

vi. 22. If four lines are proportional, the squares on them are proportional; and, conversely, if four squares are proportional, their sides are proportional.

vi. 23. (α) Triangles which have an angle of the one equal, or supplementary, to an angle of the other, have to one another the ratio compounded of the ratios of the sides about these angles.

(β) Equiangular parallelograms are in the same ratio as the rectangles under the sides forming a pair of equal angles.

(γ) Triangles which have an angle of the one equal, or supplementary, to an angle of the other, are in the same ratio as the rectangles under the sides forming these angles.

vi. 25. The shape of any given rectilineal figure may be changed (without altering its area) to the shape of any other given rectilineal figure.

vi. 30. The greater segment will be itself divided *in extreme and mean ratio* by setting off, from one end of it, a part equal to the lesser segment; and this process can be continued indefinitely. Cf. *Cor.* to ii. 11, and v. *Addenda* (1).

Some Immediate Developments of the Props. in Book vi.—not so obvious as to be properly called Corollaries.

Theorem (1)—*Triangles (or parallelograms) are to each other in the ratio compounded of the ratios of their bases and altitudes.*

Let X, X' be the areas of two $\triangle$s (or two $\square$s);
a, a' their respective altitudes;
b, b' their corresponding bases.

Then, if Y is the area of a $\triangle$ (or $\square$) of altitude a, and base b',

$$X : Y = b : b',$$

$$\text{and}\quad Y : X' = a : a'.$$

$$\therefore\quad X : X', \text{ which } = (X : Y)(Y : X'),$$

$$\text{also } = (b : b')(a : a').$$

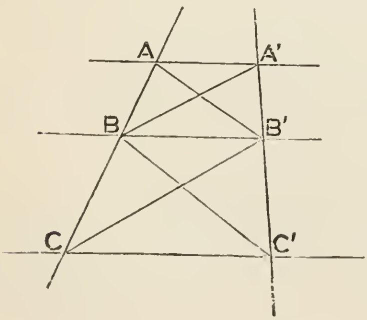
Cor. (1). If triangles (or parallelograms) have equal areas, any pair of their altitudes are *reciprocally* proportional to the bases to which they are drawn.

Cor. (2). Triangles (or parallelograms) have the same ratio to each other as rectangles under their respective altitudes and bases.

Cor. (3). Since in equiangular triangles, the altitudes are as the bases to which they are drawn, vi. 19 is an immediate deduction from the above.

Cor. (4). Similarly vi. 23 is deducible from it.

Theorem (2)—*If two lines are cut by three parallels, the intercepts on the one are in the same ratio as the corresponding intercepts on the other.*



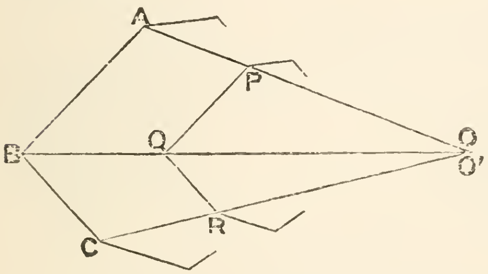
Let the three $\|$s AA', BB', CC', cut other two lines in A, B, C and A', B', C' respectively.

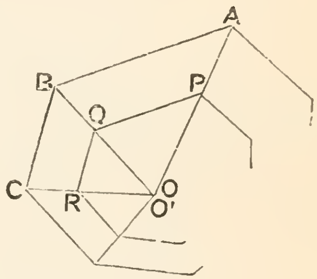
Join AB', $A'B$, BC', $B'C$.

Then $\triangle ABB' = \triangle A'BB'$;
and $\triangle CBB' = \triangle C'BB'$.
Now $AB : BC = \triangle ABB' : \triangle CBB'$,
and $\therefore = \triangle A'BB' : \triangle C'BB'$,
$= A'B' : B'C'$.

Note—This Theorem (which is here deduced from vi. 1) might have been proved directly from the definition of proportion, in the same way as vi. 1.

THEOREM (3)—*If two similar unequal rectilineal figures are so placed that their corresponding sides are parallel, then the joins of corresponding corners are all concurrent.*



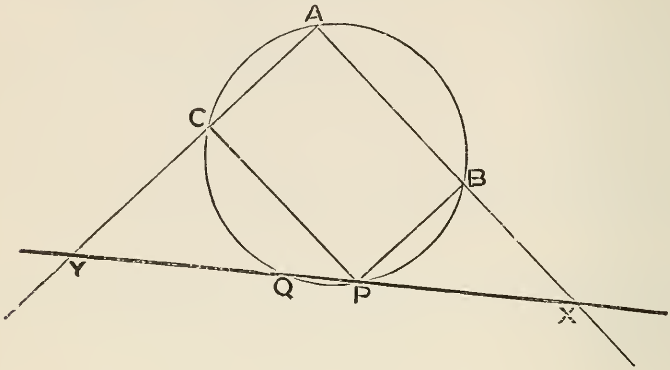


Let AB, BC be two consecutive sides of one fig.
PQ, QR the corresponding sides of the other.
Suppose that AP, BQ meet in O;
and that CR, BQ meet in O'.
Then $BO : QO = AB : PQ$,
$= BC : QR$, $\because$ figs. are simr.
$= BO' : QO'$.
$\therefore$ O and O' are the same pt.
i.e. AP, BQ, CR are concurrent.
Similarly *all* corresponding joins are concurrent in O.

Def. The point so determined is called a **centre of similarity** of the figures. Cf. p. 253.

Note—Figures like the above are said to be *similarly situated* with respect to each other.

Extension of vi. 13—*Philo's mode of finding two mean proportionals between two lines, by the use of a* **graduated ruler.**



Place the lines in positions AB, AC, at rt. $\wedge^s$.

Complete rect. ABPC ; and describe ⊙ round this rect.

Now place a *graduated* ruler, with its edge at P, and meeting AB, AC produced respectively in X, Y ; and the ⊙ in Q.

Turn the ruler about P, until PX = QY, and ∴ also QX = PY.

Then XA . XB = XP . XQ = YP . YQ = YA . YC.

And CP (or AB) : CY = AX : AY = CY : BX.

Also CY : BX = AX : AY = BX : BP (or AC);

i. e. CY is a mean proportional between AB, BX ;

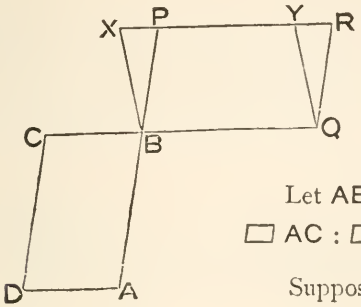
and BX „ „ CY, AC.

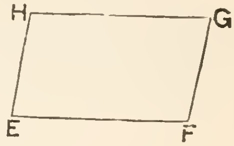
Note—A line XPQY (drawn as in fig.) through any pt. P, within *any* angle A ; so that, AQ being perpendicular to XY, then QX, PY are equal, is called **Philo's Line** : it possesses the property, the proof of which is given on p. 426, that XPY is the *least* line through P terminated by the sides of the angle.

Theorem (4)—*(Another Converse to the first part of* vi. 14 *or* 15*). If two parallelograms (or triangles) have equal areas, and the sides about a pair of angles reciprocally proportional, then the angles contained by these sides are either equal, or supplementary.*

This may be proved by drawing the altitudes to an homologous pair of the reciprocal sides ; and then using vi. *Addenda* (1) *Cor.* (1), and vi. 7.

THEOREM (5)—*(Converse of* vi. 23*). If parallelograms have to each other the ratio which is compounded of the ratios of their sides, then they are equiangular to each other.*





Let ABCD, EFGH be $\square$s such that

$\square$ AC : $\square$ FH = (AB : FG) (BC : GH).

Suppose $\hat{B}$ and $\hat{G}$ each $\not>$ a rt. $\angle$.

Produce AB to P, and CB to Q, so that BP = FG, and BQ = GH.

Complete $\square$ QBPR.

Then, since $\square$s AC, PQ are equiang.,

$\therefore$ $\square$ AC : $\square$ PQ = (AB : BP) (BC : BQ),

= (AB : FG) (BC : GH),

= $\square$ AC : $\square$ HF.

$\therefore$ $\square$ PQ = $\square$ HF.

And, since BQ = EF, $\square$s have same altitude,

$\therefore$, if $\square$ HF is applied to $\square$ PQ, so that EF coincides with BQ,

then HG must be in line with PR.

Assume that H, G are respectively at X, Y, which do *not* coincide with P, R.

Then BP = EH = BX;

$\therefore$ $B\hat{P}X = B\hat{X}P$.

But $B\hat{P}X = P\hat{B}Q = C\hat{B}A$, which $\not>$ a rt. $\angle$.

And $B\hat{X}P = E\hat{H}G$, which $\not<$ a rt. $\angle$, $\because$ $\hat{G}$ $\not>$ a rt. $\angle$.

$\therefore$ of $B\hat{P}X$ and $B\hat{X}P$, one $\not>$ a rt. $\angle$, and one $\not<$ rt. $\angle$.

And they cannot each be right.

$\therefore$ they are unequal.

But they were before proved equal.

$\therefore$ the assumption that X, Y do *not* coincide with P, R has led to a contradiction; and $\therefore$ is not true.

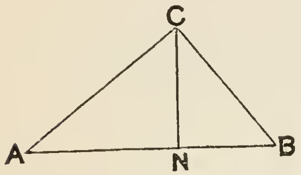
$\therefore$ $H\hat{E}F = P\hat{B}Q = C\hat{B}A$:

i. e. AC, FH are equiang. to each other.

THEOREM (6)—*If the hypotenuse of a right-angled triangle is divided in extreme and mean ratio by the altitude drawn to it; then—*

(α) *the lesser side containing the right angle is equal to the alternate segment of the hypotenuse; and conversely:*

(β) *the greater side containing the right angle is a mean proportional between the hypotenuse and the remaining side; and conversely.*



Let ABC be a $\triangle$, having $A\hat{C}B$ right; and let CN be $\perp$ from C on AB.

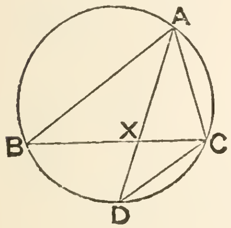
Then, by vi. 8 (γ), and vi. 17,

	$AB \cdot BN = BC^2$,
and	$AB \cdot AN = AC^2$.
$\therefore$ if (α) we assume that	$AB \cdot BN = AN^2$,
we have	$BC = AN$;
or conversely, if	$BC = AN$,
then	$AB \cdot BN = AN^2$.
And if (β) we assume that	$AB \cdot BN = AN^2$,
then by (α)	$BC = AN$,
and $\therefore$	$AB \cdot BC = AC^2$;
or conversely, if	$AB \cdot BC = AC^2$,
then	$BC = AN$,
and $\therefore$ by (α)	$AB \cdot BN = AN^2$.

Cor. From the above, a right-angled triangle can be constructed on a given line as hypotenuse, and so that its sides are in continued proportion: for if AB is the given line, then dividing it in N in extreme and mean ratio, and drawing NC perpendicular to it to meet semi-circle on AB in C, gives ABC such a triangle.

SOME USEFUL THEOREMS MAINLY DEPENDING ON BOOK vi.

THEOREM (7)—*The rectangle under two sides of a triangle is equal to the square on the line bisecting the angle between them, and terminated by the opposite side, together with the rectangle under the segments into which the third side is divided by that bisector.*



In $\triangle$ ABC let AX, which bisects $\widehat{BAC}$, meet BC in X.

Produce AX to meet the circum-⊙ of $\triangle$ ABC in D; and join DC.

Then since $\widehat{ABX} = \widehat{ADC}$, in same segt,

and $\widehat{BAX} = \widehat{DAC}$;

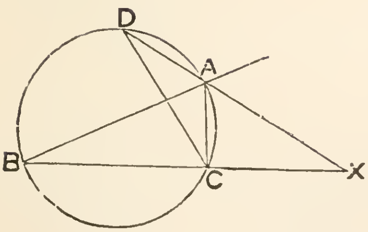
$\therefore$ $\triangle$ AXB is equiang. to $\triangle$ ACD.

$\therefore$ BA : AX = DA : AC.

$\therefore$ BA . AC = DA . AX,

$= AX^2 + DX . AX,$

$= AX^2 + BX . XC.$



All the foregoing holds for the ext. bisector (see adjacent fig.) excepting that, in the last two lines AX^2 has to be subtracted, so that we get

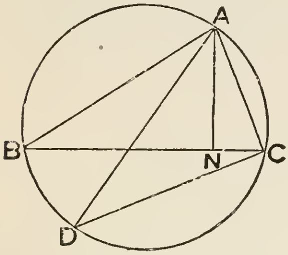
$$BA . AC = BX . XC - AX^2.$$

Also if AX is a tang. to the ⊙, it is ∥ to BC, and Prop. has no meaning.

Cor. BD^2 (or CD^2) = AD . DX;

for BD : DA = CX : CA = DX : BD.

THEOREM (8)—*The rectangle under two sides of a triangle is equal to the rectangle under the altitude conterminous with them, and the diameter of the circum-circle.*



Let AD be diam. of ⊙ round △ ABC; and AN the altitude from A.

Join DC.

Then since $A\hat{B}C = A\hat{D}C$, in same segt.

and $A\hat{N}B = A\hat{C}D$, each being right;

∴ △ ABN is equiang. to △ ADC.

∴ BA : AN = AD : AC.

∴ BA . AC = AN . AD.

Cor. (1). The rectangles under any two sides of any triangles, inscribed in the same or equal circles, are as the altitudes drawn to the third sides.

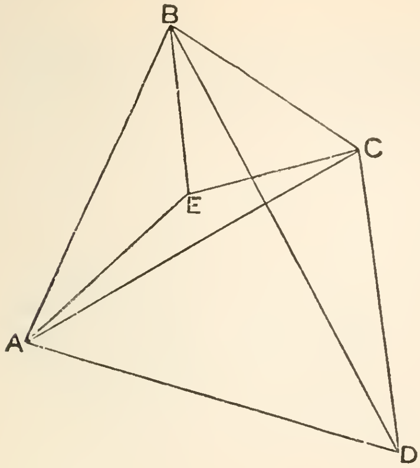
Cor. (2). If ABCD is a cyclic quadrilateral, whose diagonals cut in O, then—

AB . BC : CD . DA = BO : DO;

and AB . BC : BC . CD = BO : CO.

Note—The *converses* of all the above theorems easily follow.

THEOREM (9)—*The rectangle under the diagonals of any quadrilateral is less than the sum of the rectangles under its opposite sides—excepting in the particular case of a cyclic quadrilateral, when the inequality becomes an equality.*



Let ABCD be a quad. *not* cyclic, so that $\wedge^s$ BAC, BDC are *not* equal.

Draw AE, BE, within the quad, so that

$$B\hat{A}E = B\hat{D}C, \text{ and } A\hat{B}E = C\hat{B}D.$$

Join EC.

Then $\triangle^s$ AEB, DCB are simr. by construction.

$$\therefore \quad AB : AE = DB : DC;$$

$$\therefore \quad AB \,.\, CD = DB \,.\, AE \quad . \; . \; . \; . \; . \; . \; . \; . \quad (\alpha)$$

Again, since $A\hat{B}D = E\hat{B}C$,

and $AB : BD = EB : BC$;

$\therefore$ $\triangle^s$ ABD, EBC are simr.

$$\therefore \quad AD : DB = EC : BC;$$

$$\therefore \quad AD \,.\, BC = DB \,.\, EC \quad . \; . \; . \; . \; . \; . \; . \; . \quad (\beta)$$

Adding results (α) and (β) we get

$$AB \,.\, CD + AD \,.\, BC = DB\,(AE + EC);$$

$$\therefore \quad > DB \,.\, AC.$$

In the particular case when A, B, C, D are concyclic, E will lie in AC; so that then AE + EC = AC, and the inequality becomes an equality.

Note (1)—The particular case is known as *Ptolemy's Theorem*: it is one of the most useful properties of the circle. The Student should make out (which he will find easy) an independent proof of this case.

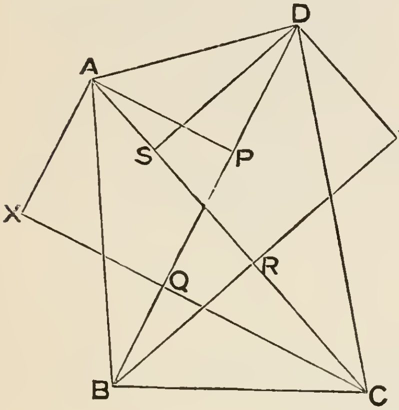
Note (2)—Of course it follows from the preceding that, if

$$AB \,.\, CD + AD \,.\, BC = AC \,.\, BD,$$

then ABCD is a cyclic quadrilateral. This is the *converse* of *Ptolemy's Theorem*.

Note (3)—Of the three rects. AB . CD, AD . BC, AC . BD, the sum of any two > the third, unless ABCD is cyclic; for they are propl. to sides of $\triangle$ AEC.

THEOREM (10)—*The diagonals of a cyclic quadrilateral are proportional to the sums of the rectangles under the sides meeting at their respective extremities.*



1°, let ABCD be *any* quad.; AP, CQ $\perp^s$ to diag. BD; and BR, DS $\perp^s$ to diag. AC.

Draw DY $\perp$ to BR produced, and AX $\perp$ to CQ produced.

$$\text{Then}\quad R\hat{C}Q = R\hat{B}Q, \because\ RCBQ \text{ is cyclic.}$$

And as these are $\wedge^s$ of $\triangle^s$ ACX, DBY, in which also $\hat{X}$ and $\hat{Y}$ are each right;

$$\therefore\quad \triangle\ AXC \text{ is equiang. to } \triangle\ DYB.$$

$$\therefore\quad AC : BD = CX : BY,$$
$$= AP + CQ : BR + DS.$$

2°, let ABCD be supposed cyclic, and δ the diam. of its circum-$\odot$, then

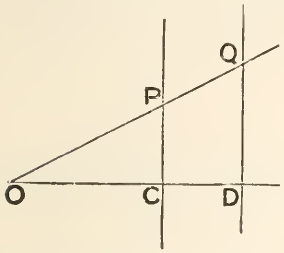
$$AB \,.\, AD = AP \,.\, \delta,$$
$$CB \,.\, CD = CQ \,.\, \delta,$$
$$BA \,.\, BC = BR \,.\, \delta,$$
$$DA \,.\, DC = DS \,.\, \delta;$$

$$\therefore\quad AB \,.\, AD + CB \,.\, CD : BA \,.\, BC + DA \,.\, DC$$
$$= (AP + CQ)\,\delta : (BR + DS)\,\delta,$$
$$= AC : BD.$$

Note (1)—The *converse* of the above Theorem will readily follow.

Note (2)—If **a**, **b**, **c**, **d** are the sides; **x**, **y** the diags. of a cyclic quad.;
then, by above, $\mathbf{x} : \mathbf{y} = \mathbf{ad} + \mathbf{bc} : \mathbf{ab} + \mathbf{cd}$,
and, by Ptolemy, $\mathbf{xy} = \mathbf{ac} + \mathbf{bd}$;
$\therefore$ **x**, **y** are known, if **a**, **b**, **c**, **d** are given.

THEOREM (11)—*If from a fixed point* O *a variable line is drawn, and in it points* P, Q *are taken, so that the ratio of* OP *to* OQ *is constant; then, if the Locus of one of the points is a line, so also is the Locus of the other.*



Let Locus of P be a line.

Draw OC ⊥ to it; and QD ∥ to PC, to meet OC in D.

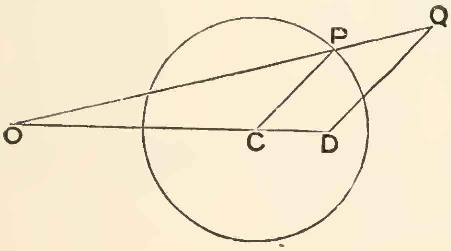
Then OD : OC = OQ : OP, const.

And OC is const.

∴ OD is const.

∴ Locus of Q is fixed line QD.

THEOREM (12)—*If from a fixed point* O *a variable line is drawn, and points* P, Q *are taken in it, so that the ratio of* OP *to* OQ *is constant; then, if the Locus of one of the points is a circle, so also is the Locus of the other.*



Let Locus of P be a ⊙, centre C.

Draw QD ∥ to CP, to meet OC in D.

Then DQ : CP = OQ : OP, const.

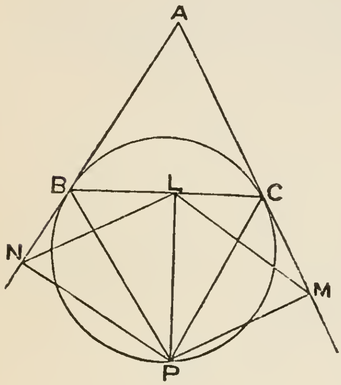
And CP is const.

∴ DQ is const.

∴ Locus of Q is fixed ⊙, centre D, radius DQ.

Note—A useful particular case, when P is the mid point of OQ, has been given already (p. 190, Exercise 109).

THEOREM (13)—*The Locus of a point such that the rectangle under its distances from the equal sides of an isosceles triangle is equal to the square on its distance from the third side, is the circle which touches the equal sides at the extremities of the third side.*



Let ABC be an isos. $\triangle$, in which

$$AB = AC.$$

Let P be a pt. such that, if L, M, N are the respective feet of $\perp^s$ from it on BC, CA, AB,

then $PM \cdot PN = PL^2$.

Join LM, LN, PB, PC.

Then PLCM and PLBN are cyclic quads.

$$\therefore \quad L\hat{P}M = A\hat{C}B = A\hat{B}C = L\hat{P}N.$$

And $PM : PL = PL : PN$.

$\therefore \quad \triangle^s$ MPL, LPN are simr.

$$\therefore \quad M\hat{L}P = L\hat{N}P;$$

$$\therefore \quad P\hat{B}C = L\hat{N}P = M\hat{L}P = M\hat{C}P.$$

$\therefore \quad \odot$ round BPC has BC a secant and AC a tang. at C.

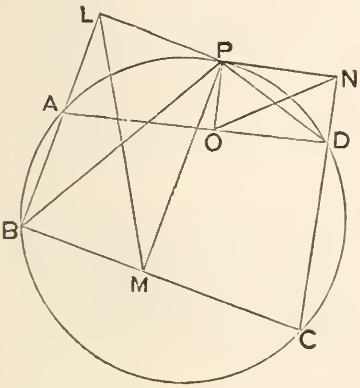
Similarly it has AB a tang. at B.

$\therefore \quad$ it is a fixed $\odot$:

i.e. this $\odot$ is Locus of P.

Note—The *converse* of this is easily proved in a similar manner.

THEOREM (14)—*The Locus of a point such that the rectangle under its distances from an opposite pair of sides of a cyclic quadrilateral is equal to the rectangle under its distances from the other opposite pair, is the circle which circumscribes the quadrilateral.*



Let ABCD be a cyclic quad.; PL, PM, PN, PO the respective $\perp^s$ on AB, BC, CD, DA from a pt. P, such that

$$PL . PN = PM . PO.$$

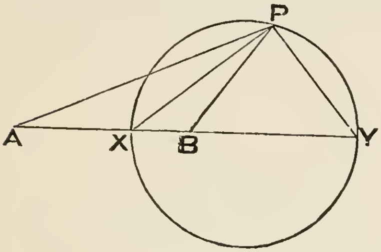
Join LM, NO, PB, PD.

Then $\widehat{LPM}$ = suppt. $\widehat{ABM}$, since LBMP is cyclic,
= $\widehat{ADC}$, since ABCD is cyclic,
= $\widehat{NPO}$, since NPOD is cyclic,

Also PL : PM = PO : PN ;
∴ △ LPM is equiang. to △ OPN.
∴ $\widehat{LMP} = \widehat{ONP}$.
But $\widehat{LMP} = \widehat{LBP}$, ∵ LBMP is cyclic.
And $\widehat{ONP} = \widehat{PDO}$, ∵ PODN is cyclic.
∴ $\widehat{ABP} = \widehat{PDA}$.
∴ P is concyclic with ABD :
i. e. Locus of P is ⊙ round ABCD.

Note—The *converse* of this is easily proved in a similar manner.

THEOREM (15)—*If* A, B *are fixed points, and* P *a variable point, such that the ratio of* PA *to* PB *is one of constant inequality; then, if* AB *is divided internally in* X, *and externally in* Y, *in the same ratio, the circle on* XY *as diameter is the Locus of* P. (*Apollonius' Locus*)



For, since AX : BX = AP : BP,

$\therefore$ PX is int. bisector of $A\hat{P}B$.

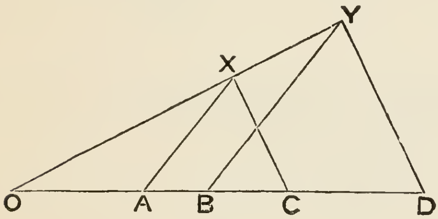
And since AY : BY = AP : BP,

$\therefore$ PY is ext. bisector of $A\hat{P}B$.

$\therefore$ $X\hat{P}Y$ is right.

$\therefore$ P is on ⊙ diam. XY.

THEOREM (16)—*If* A, B, C, D *are collinear points, and on* AC, BD *any similar triangles* AXC, BYD *are described, so that the homologous sides* AX, BY *are parallel, as also* CX, DY; *and if* O *is the intersection of* YX, DA, *then the rectangle under* OA . OD *is equal to the rectangle under* OB . OC.

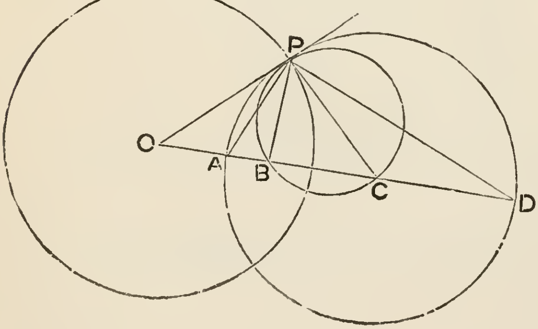


For OA : OB = OX : OY,

= OC : OD.

$\therefore$ OA . OD = OB . OC.

THEOREM (17)—*If* O, A, B, C, D, *are points situated as in the last Theorem; then the circle whose centre is* O, *and the square on whose radius is equal to either of the equal rectangles, is the Locus of the point at which* AB, CD *subtend equal angles.*



Let ⊙, centre O, have radius OP such that

OA . OD = OP^2 = OB . OC.

Describe ⊙s round BPC and APD.

Then, by the above condn., OP touches both these ⊙s.

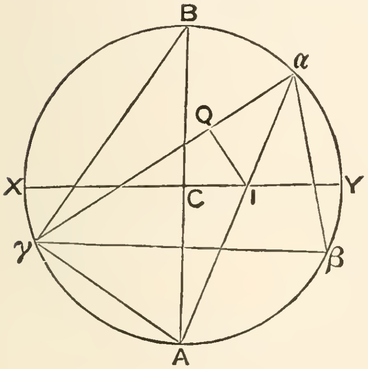
$$\therefore\ \widehat{APB} = \widehat{OPB} - \widehat{OPA},$$
$$= \widehat{PCB} - \widehat{PDA},$$
$$= \widehat{CPD}.$$

i. e. every pt. on ⊙, radius OP, subtends equal $\wedge^s$ at AB, CD.

Cor. When PB, PC coalesce, this Locus is the same as that in (15).

THEOREM (18)—*(Chapple's) If* R, r *are the respective radii of the circum-circle and in-circle of a triangle, whose corresponding centres are* C *and* I; *then*

$$CI^2 = R^2 - 2\,Rr.$$



Let $\alpha\beta\gamma$ be the △.

Let αI meet the circum-⊙ in A; and AC meet same ⊙ in B.

Join Bγ, Aγ.

Draw IQ ⊥ to $\alpha\gamma$; and let CI meet the ⊙ in X, Y.

Then $\widehat{AB\gamma} = \widehat{I\alpha Q}$, in same segt.

And $\widehat{A\gamma B} = \widehat{IQ\alpha}$, since each is right.

$\therefore$ $\triangle^s$ ABγ, IαQ are simr.

$$\therefore\ I\alpha : IQ = AB : A\gamma.$$

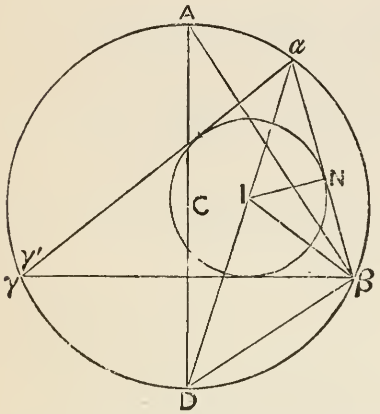
But IQ = r, AB = 2 R, and Aγ = AI. [See p. 212, *Cor.* (3).]

$$\therefore\ 2\,Rr = AI \,.\, I\alpha,$$
$$= XI \,.\, IY,$$
$$= (R + CI)(R - CI),$$
$$= R^2 - CI^2:$$

i.e. $CI^2 = R^2 - 2\,R\,r.$

Note—In the same manner, if E_1 is the ex-centre of the ⊙ whose radius is r_1 it can be proved that $CE_1^2 = R^2 + 2\,R\,r_1$.

THEOREM (19)—*If* C *and* I *are the centres of two circles, whose respective radii are* R, r, *and which are so situated that* $CI^2 = R^2 - 2Rr$; *and if from any point* α, *on the outer, tangents* $\alpha\beta$, $\alpha\gamma$ *are drawn to the inner, so that* β, γ *are concyclic with* α; *then will* $\beta\gamma$ *also be a tangent to the inner.*



Let tang. from β to the inner ⊙ meet $\alpha\gamma$ in γ': we have to show that γ and γ' are the same pt.

Let N be pt. of cont. of $\alpha\beta$; and let αI meet outer ⊙ in D.

Draw diam. DCA; and join Aβ, Dβ.

$$\begin{aligned}
\text{Then}\quad 2Rr &= R^2 - CI^2,\\
&= (R + CI)(R - CI),\\
&= \alpha I \cdot ID.\\
\therefore\quad IN : I\alpha &= ID : DA.\\
\text{Again, since } \widehat{DA\beta} &= \widehat{D\alpha\beta}, \text{ in same segt.}\\
\text{and}\quad \widehat{D\beta A} &= \widehat{IN\alpha}, \text{ each being right;}\\
\therefore\quad \triangle D\beta A &\text{ is equiang. to } \triangle IN\alpha.\\
\therefore\quad IN : I\alpha &= D\beta : DA.\\
\therefore\quad ID &= D\beta.\\
\therefore\quad \widehat{DI\beta} &= \widehat{D\beta I}.\\
\text{Now}\quad \widehat{D\beta\gamma} &= \widehat{D\alpha\gamma},\\
&= \widehat{I\alpha\beta},\\
&= \widehat{DI\beta} - \widehat{I\beta\alpha},\\
&= \widehat{D\beta I} - \widehat{I\beta\gamma'},\\
&= \widehat{D\beta\gamma'}.\\
\therefore\quad \gamma \text{ and } \gamma' &\text{ are the same pt.*}
\end{aligned}$$

* BESANT—*Quarterly Journal of Mathematics*, Vol. XII.

Note (1)—Hence the problem—*To describe a triangle which shall be inscribed in circle (radius* R*) and circumscribed about circle (radius* r*)*—is impossible, unless the square on the distance between the centres of the circles $= R^2 - 2Rr$; and, if that is the case, an infinite number of such triangles can be described: in this latter case the Problem is said to be *indeterminate*.

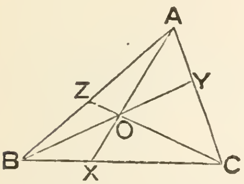
Def. A problem whose solution is impossible unless a certain condition is satisfied, and which, when that condition is satisfied is indeterminate, is called a PORISM.

As Examples: Problem 34 on p. 406; and, when n is even, Problem 51 on p. 408; are Porisms.

Note (2)—The generalisation of this Theorem for any polygon is called *Poncelet's Theorem*. See Exercise 45 on page 417.

THEOREM (20)—*If through any point* O, *within a triangle* ABC, *lines* AX, BY, CZ *are drawn from* A, B, C *to meet the respectively opposite sides in* X, Y, Z; *then*

$$\triangle AOB : \triangle AOC = BX : CX.$$



For $\triangle AOB : \triangle BOX = AO : OX,$

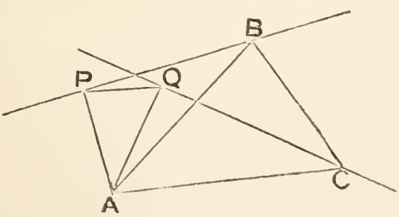
$= \triangle AOC : \triangle COX.$

$\therefore \quad \triangle AOB : \triangle AOC = \triangle BOX : \triangle COX,$

$= BX : CX.$

Def. A rectilineal figure is said to be of **given species**, when its angles, and the ratios of the sides forming them, are given.

THEOREM (21)—*If a triangle of given species has one corner fixed, and another corner always on a fixed line; then the third corner will always be on a fixed line.*



Let ABC be a $\triangle$ of given species, which turns round A, so that B is always on fixed line BP.

Draw AP $\perp$ to BP; and PQ, AQ, so that $\widehat{APQ} = \widehat{ABC}$,

and $\widehat{PAQ} = \widehat{BAC}$.

Join CQ, and produce it both ways.

Then, by the construction, $\triangle^s$ APQ, ABC are simr.

$\therefore$ CA : AQ = BA : AP.

But $\widehat{CAQ} = \widehat{BAP}$.

$\therefore$ $\triangle^s$ CAQ, BAP are simr.

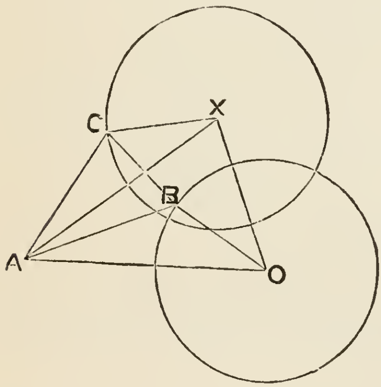
$\therefore$ $\widehat{CQA} = \widehat{BPA}$, a rt. $\wedge$.

But Q is a fixed pt., and QA a fixed direction,

$\therefore$ CQ is a fixed direction:

i. e. C is always on a fixed line.

THEOREM (22)—*If a triangle, of given species, has one corner fixed, and another corner always on a fixed circle; then the third corner will always be on a fixed circle.*



Let ABC be a $\triangle$ of given species, which turns round A, so that B is always on $\odot$, centre O.

Join OA; and draw OX, AX,

so that $\widehat{AOX} = \widehat{ABC}$,

and $\widehat{OAX} = \widehat{BAC}$.

Then, by the construction, $\triangle^s$ AOX, ABC are simr.

$\therefore$ AO : AB = AX : AC.

But $\widehat{OAB} = \widehat{XAC}$.

$\therefore$ $\triangle^s$ OAB, XAC are simr.

$\therefore$ AO : OB = AX : XC;

i. e. XC is a fourth propl. to three fixed lengths;

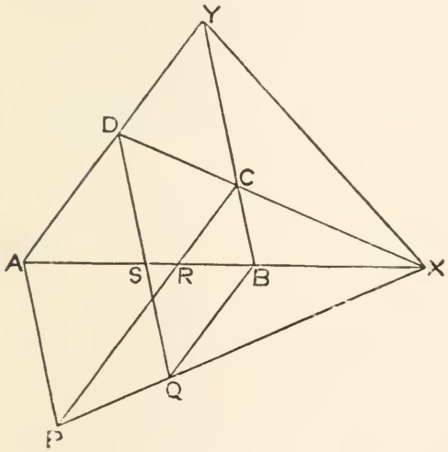
and it is drawn from a fixed pt. X:

i. e. C is on fixed $\odot$, centre X, radius CX.

Def. If the opposite pairs of sides of a quadrilateral (defined as on p. 51) are produced to meet, and their points of intersection joined; then the join is called the **third diagonal** of the quadrilateral; and the figure thus formed is called a **complete quadrilateral.**

Note—The term *complete quadrilateral* is sometimes used in the following more extended sense—Let there be four indefinite lines, of which no three pass through the same point: they will enclose a four-sided figure (the ordinary quadrilateral) and will have six points of intersection, four of the adjacent sides of the figure, and two others of opposite sides: these six points form three pairs of opposite corners; and there will be three joins of these opposite corners, intersecting in three points. Then the entire figure, consisting of seven lines, intersecting in nine points, is called a **complete quadrilateral**; and the three joins of the pairs of opposite corners are called its *diagonals.*

THEOREM (23)—*The mid points of the three diagonals of a complete quadrilateral are in one line.*



In quad. ABCD, let AB, DC meet in X; and BC, AD in Y; so that XY is the third diag.

Complete the ▱s CYAP, BYDQ.

Let CP, DQ cut AX in R, S respectively.

Join PQ, QX.

Then XB : BS = XC : CD = XR : RA.

∴ XB : XR = BS : RA = BQ : RP.

∴ P, Q, X are in one line.

∴ mid pts. of YP, YQ, YX are in a ∥ to PQX.

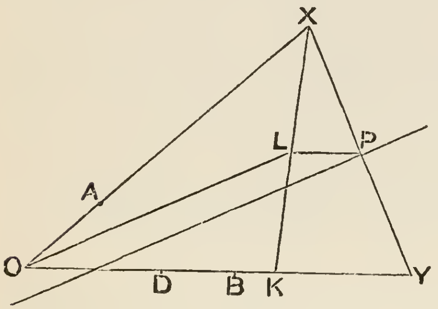
But mid pt. of YP is mid pt. of AC, ∵ they are diags. of ▱ CYAP.

And mid pt. of YQ is mid pt. of BD, ∵ they are diags. of ▱ BYDQ.

i.e. the mid pts. of the three diags. AC, BD, XY are in one line.*

* Taken from *Taylor's Ancient and Modern Geometry of Conics* (p. 255), by permission of the Author.

THEOREM (24)—*(Newton—Principia, Book* i. *Lemma* 23*) If from two fixed points* A, B, *lines* AX, BY *are drawn in fixed directions, and so that* AX *to* BY *is a fixed ratio ; and if* P *is taken in* XY *so that* PX *to* PY *is a fixed ratio ; then the Locus of* P *is a fixed line.*



Let XA, YB meet in O.

Take D in BO, so that

AX : BY = AO : BD.

Take K in YO, so that

YK = OD.

Then since AX : AO = BY : BD ;

∴ OX : YD (or OK) = AX : BY ;

∴ species of △ XOK is fixed.

Draw PL ‖ to OY, to meet XK in L ; and join OL.

Then, since OK : KX is fixed ;

and KX : KL is also fixed ;

∴ OK : KL is fixed.

∴ species of △ OKL is fixed.

∴ OL is fixed in direction.

But LP is fixed in length and direction.

∴ Locus of P is a fixed line ‖ to OL.

EXERCISES ON BOOK vi.

Note—*These Exercises are all Theorems to be proved; and depend mainly on the principles of Book* vi.

1. If from a point outside a circle, a pair of tangents and a secant are drawn, the quadrilateral formed by joining the points of section to the points of contact, has the rectangles under its opposite sides equal.

2. If two circles touch, a common tangent is a mean proportional between their diameters.

3. CAB, DAB are two triangles on same side of AB; if P is any point in AB; and PX, PY parallels to AC, AD, meet BC, BD respectively in X, Y; then XY is parallel to CD.

4. The diagonals of a regular pentagon cut each other in extreme and mean ratio.

5. If a radius of a circle is cut in extreme and mean ratio, the greater segment is equal to a side of a regular inscribed decagon.

6. The following group of Theorems are all *deducible* from *Ptolemy's Theorem*—vi. *Addenda* (9) *Note* (1)—

(1) The distance of any point on the circum-circle of an equilateral triangle, from the farthest corner, is equal to the sum of its distances from the other two corners.

(2) If the diagonals of a cyclic quadrilateral cut at right angles, then the rectangles under the opposite sides are together double the area of the quadrilateral.

(3) A, B are fixed points on the circumference of a circle; if P is a variable point on the same circumference, and C the mid point of the arc AB, then the ratio PA + PB to PC is constant.

(4) ABCDE is a regular pentagon; if P is any point on the arc AE of its circum circle, then PA + PC + PE = PB + PD.

Note—*Apply the Theorem to quads.* PABC, PBCD, PBCE.

(5) A similar Theorem to (4) holds for a regular heptagon.

(6) A variable circle goes through the vertex A of a fixed angle, and cuts its sides in X, Y; if the circle also goes through a second fixed point B, then—

$$l \cdot AX + m \cdot AY = n \cdot AB;$$

where l, m, n are constants whose ratio is determinable.

(7) ABCD is a parallelogram; if a variable circle through A cuts AB, AC, AD in X, Y, Z respectively, then—

$$AX \cdot AB + AZ \cdot AD = AY \cdot AC.$$

7. PR, QS are fixed lines; if OPQ, ORS are variable lines (cutting them) but always parallel to fixed directions; then the ratio of OP . OQ to OR . OS is constant.

8. If from any point P tangents PA, PB are drawn to a circle, and AC is drawn perpendicular to the diameter BD; then AC is bisected by PD.

9. If two regular polygons (of the same number of sides) are, one inscribed in, and the other circumscribed about the same circle; and if another polygon (of double the number of sides) is inscribed in the circle; then the area of the latter polygon is a mean proportional between the areas of the two former.

NOTE—*If* C *is cent. of* ⊙ *;* CA *a radius, and* AB *a tang. whose length* = $\frac{1}{2}$ *side of outer pol.; then, drawing* DN ⊥ *to* CA, *from* D, *where* CB *cuts* ⊙, *the pols. are propl. to* △s CDN, CDA, CBA; and result follows by vi. 1, 4, 19.

10. ABC is a triangle; AX, BY, CZ are drawn to meet the opposite sides, and to be equal; if from any point P, within the triangle, PL, PM, PN are drawn to meet the sides, and be parallel to AX, BY, CZ; then—

$$PL + PM + PN = AX.$$

NOTE—△ BPC : △ ABC = PL : AX; *and simr. results.*

11. The converse of vi. 19 is true, if the triangles are assumed isosceles, and the bases are taken as *the* pair of sides such that their duplicate ratio is equal to the ratio of the areas of the triangles.

12. P is a point in AB, and Q in AC, of triangle ABC, such that BP, CQ are equal; if PQ, BC, when produced, meet in R, then—

$$AC : AB = PR : QR.$$

13. If X is any point in the side BC (or BC produced) of a triangle ABC; then—

radius ⊙ round ABX : radius ⊙ round ACX = AB : AC.

14. If P is any point in median AM, of triangle ABC; and BP, CP meet AC, AB in X, Y; then XY is parallel to BC.

15. The last Exercise gives a means of drawing a parallel, through a given point, to a given finite line, the position of whose mid point is given, by means of a ruler only.

16. If C is the centre of a fixed circle, a tangent to which meets another fixed circle (through C) in P, Q; then CP . CQ is constant.

17. On a level plain are to be seen two church spires: a person walking on the plain, so as always to see the spires at equal angles of elevation, will walk in a circle.

18. If one triangle is so inscribed in another, that each pair of sides of the inner make equal angles with that side of the outer at which they are conterminous; then the inner is the pedal triangle of the outer.

NOTE – *Converse of* iii. *Addenda* (19). *Use* vi. 3, *for external bisection.*

19. From a corner A, of a triangle ABC, AX, AY are drawn to the opposite side, so that the angles BAX, CAY are equal: prove that

$$BX \cdot BY : CX \cdot CY = AB^2 : AC^2.$$

NOTE—*If* ⊙ *round* AXY *meets* AB, AC *in* P, Q; *then* PQ *is* ∥ *to* BC.

20. If two sides of a triangle are unequal, the sum of the greater side and the perpendicular upon it from the opposite corner is greater than the sum of the lesser side and the perpendicular upon it from the opposite corner.

21. Two circles have internal contact at P; if two perpendiculars to their line of centres meet the outer circle in A, B and the inner in C, D; then—

$$PA : PB = PC : PD.$$

22. In the figure of iv. 10, if I is the in-centre of triangle ABC; then AB is a mean proportional between BI and the perimeter of ABC.

23. If one corner of a rectangle is fixed, and the two adjacent corners move on the same fixed circle; then the fourth corner moves on a fixed circle concentric with the other.

24. ABC is a triangle; if P is any point in BC, and PX, PY are parallel to AC, AB and meet AB, AC in X, Y respectively; then the triangle AXY is a mean proportional between the triangles BPX, CPY.

25. From the corners of a parallelogram perpendiculars are dropped on the diagonals: show that the joins of the feet of these perpendiculars form a parallelogram similar to the original one.

26. From any point within a parallelogram perpendiculars are dropped on its sides: show that the area formed by the joins of their feet is constant.

27. A line is divided into two parts in the ratio of 3 to 1, and on each of these parts as diameter a circle is described; if a common tangent to these circles is drawn, it meets the common diameter line at a distance from the lesser circle which is equal to its radius.

28. Two variable circles cut at a fixed point, and have their centres on two fixed lines, also cutting at that point; show that their common tangents meet on one of two fixed lines through the fixed point.

29. If a line is drawn to cut two intersecting circles, and their common chord, the successive points of section being A, B, C, D, E; then—

1°, $AB : BC = ED : DC$;

2°, $AE^2 : BD^2 = AC \cdot CE : BC \cdot CD$.

30. If AB is any chord of a circle (centre C) and AP, BP are drawn to any point P on the circumference, and cut the diameter perpendicular to AB in X, Y; then CX . CY = $(\text{radius})^2$.

31. BAC is a triangle, right-angled at A, and AP is perpendicular to BC; if on BP, CP semi-circles are described, cutting BA, CA in X, Y respectively,

then BX : CY = the triplicate ratio of BA to CA.

32. If ABCD is a quadrilateral, and any transversal (see Index) is drawn, cutting AB, AD, BC, DC, CA, BD in a, b, c, d, e, f, respectively; then—

$$ab : cd = af \,.\, be : cf \,.\, de.$$

NOTE—*Thro.* D *draw a* || *to the transversal.*

33. If in sides AB, AC (of a triangle) M, N are respectively taken; and in MN, P is taken, so that BM : AM = AN : NC = PM : PN, then area BPC is twice area AMN.

34. XPY is a tangent to a fixed circle, at a fixed point P; PQ is a diameter; QX, QY cut the circle at U, V; and UV cuts PQ in R: if the rectangle under PX, PY is constant, then R is fixed.

35. On the diameter of a circle two equal circles are described, so that the diameter of each is a radius of the original circle; in either of the spaces between the three circles another circle is inscribed; prove that—

diam. last ⊙ : diam. either of equal ⊙s = 2 : 3.

36. If two fixed parallel tangents to a circle are cut by a variable tangent, the rectangle under the segments of the variable tangent is constant.

37. In the figure of iv. 10, if PQ, parallel to BC, meets AC in Q, then—

$$\triangle APQ : \text{fig. } PBCQ = BC : BA.$$

38. In a triangle ABC, the side BC and angle A are fixed; if the bisector of A meets BC in P, and is produced to Q, so that AP . AQ = AB . AC, then Q is a fixed point.

39. Any point P, on the circumference of a circle, is joined to A, B, the ends of a diameter; if the perpendicular to AB, at any point Q, meets the circle in X, and PA, PB in Y, Z; then $QX^2 = QY \,.\, QZ$.

40. If LMN is *Simson's Line* relatively to a point P and a triangle ABC (cf. p. 172, *Note*) then—

$$PA \,.\, PL = PB \,.\, PM = PC \,.\, PN.$$

And if X is the foot of the perpendicular from P on LMN, then—

$$PX \,.\, PA = PM \,.\, PN; \quad PX \,.\, PB = PL \,.\, PN; \quad PX \,.\, PC = PL \,.\, PM.$$

41. A circle has its centre E at the mid point of the base BC of an isosceles triangle, and touches the sides AB, AC; if a variable tangent to the circle meets AB, AC in X, Y; then $BX \cdot CY = BE^2$.

42. If in a triangle ABC, AX is perpendicular to BC, and XP, XQ to AB, AC; then, AD being a diameter of the circum-circle of ABC—

1°, $PQ \cdot AD = 2$ area $\triangle$ ABC; and 2°, PQ is $\perp$ to AD.

43. Perpendiculars from the intersection of two opposite sides of a cyclic quadrilateral on the other sides, are in the same ratio as the latter sides.

44. If M, N, X are the points in the side BC, of a triangle ABC, such that AM, AN, AX are respectively a median, an altitude, and the bisector of the angle BAC; then $(CA \sim CB)^2 = 4\, MN \cdot MX$.

45. If tangents from a point O to a circle are bisected by a line which meets any chord PQ of the circle in R, then the angles ROP, RQO are equal.

46. ABC is a triangle, right-angled at C; if CX is drawn to meet AB, and CY to meet AB produced, so that each of the angles BCX, BCY is equal to angle A, then—

$$XA : XB = \text{the duplicate ratio of } YA \text{ to } YC.$$

47. ABCD is a cyclic quadrilateral; BA, CD meet in X, and CA, BD in Y; if XY cuts AD in Z, then—

$$AZ : DZ = XA \cdot AB : XD \cdot DC.$$

48. OA, OB are lines fixed in position, and AB such that triangle AOB is of fixed area; if through P, the mid point of AB, XPY is drawn, parallel to a fixed direction, to meet OA, OB in X, Y, then $PX \cdot PY$ is constant.

49. ABCD is a quadrilateral *not* cyclic; if the ratios of the three rectangles under AB, CD, under AC, BD and under AD, BC, to one another are fixed, then the difference of any pair of angles, subtended by any one of the six lines, is also fixed.

NOTE—*Construct as in* vi. *Addenda* (9) *and show that species of* $\triangle$ ACE is *fixed.*

50. OA, OB are two lines fixed in direction; P, Q are variable points in OA, OB; PX, QY are perpendiculars on OB, OA; if PQ always goes through one fixed point, XY always goes through another.

51. Given the base of a triangle, and the difference of its base angles; if through the mid point of the base two lines are drawn parallel to the internal and external bisectors of the vertical angle, then these lines, with the bisectors, will form a rectangle of constant area.

NOTE—*See* i. *Addenda* (10) and (11).

52. Given the vertical angle and area of a triangle, show that the difference between the square on the median from the vertex, and the square on half the base, is constant.

53. In any triangle ABC, if D, E are points in BC, CA such that BD is one-fourth BC, and CE one-fourth CA; and if AD, BE cut in X, then CX produced will divide AB in the ratio of 9 to 1.

54. ABCD is a cyclic quadrilateral; if AB, DC meet in P; and BC, AD in Q, then—

sq. on PQ = sq. on tang. from P + sq. on tang. from Q.

NOTE—*Draw* BX *to meet* PQ, *so that* $\widehat{PBX} = \widehat{PQA}$; *then, by simr.* $\triangle^s$ PBX, PQA, *we have* PA . PB = PQ . PX; *or sq. on tang. from* P = PQ . PX: *then join* CX.

55. In the last Exercise, show that the circle on PQ as diameter cuts the circle round the quadrilateral orthogonally.

56. From a point A, on the outer of two concentric circles, tangents AP, AQ are drawn to the inner; if AP, QP produced, meet the outer in T, R respectively; then—

$$RP : RQ = RT^2 : RA^2.$$

57. AM is a median of a triangle ABC; AN bisects the angle BAC, and meets BC in N; if the perpendicular from B on AN meets AM in P, then PN is parallel to AB.

58. ABCD is a cyclic quadrilateral, and M the mid point of CD; if AD, BC meet in P; and PM meets AB in X; then—

$$XA : XB = PA^2 : PB^2.$$

59. Two circles cut at A, B; if P is a variable point on one circle, and PA, PB meet the other in X, Y; then the Envelope of XY is a circle.

NOTE—*If a variable line (straight or curved) moving according to some law, touches in every position a figure of any form, such figure is termed the* **Envelope** *of the line.*

60. The sides and angles of a triangle are given, but its position varies subject to the condition that two of its sides go through two fixed points; show that the Envelope of the third side is a circle.

61. Two circles cut in A, B; if X, Y are variable points on their circumferences, but such that the angle XAY is constant; and if XY is cut in a constant ratio in P, then the Locus of P is a circle.

NOTE—*Draw* XM, YN *respectively* ∥ *to* YA, XA, *and meeting* ⊙ˢ *in* M, N: *let* ∥ˢ *from* P *to* AX, AY *meet* AM, AN *in* U, V: *then it can be proved that* UV *and* $\widehat{UPV}$ *are fixed.*

62. If a parallel to the side AB, of a triangle ABC, meets CA in X, and CB in Y; then the Locus of the intersection of the circles round the triangles CAY, CBX, is a line through C.

63. If a corner of a triangle is joined to the point of contact of the in-circle (or an ex-circle) with the side opposite; then the mid point of this join, and the mid point of the side, are collinear with the centre of the circle.

64. In the figure of vi. *Addenda* (7) if the triangle varies subject to the conditions that BC is fixed, and BA + AC is constant; then AX : XD is constant.

65. In Exercise 100, page 188, show that OA : OX = OY : OC.

66. In a triangle ABC, M is the mid point of BC, D the point where the bisector of angle BAC (internal or external) meets BC; then, if MX, BY, CZ, are perpendiculars on the other bisector (external or internal) of angle BAC—BY . CZ = MX . AD.

67. Four rods PA, PB, QAC, QBD are pivoted at P, Q, A, B, so as to be capable of angular motion in one plane; and so that PAQB is a parallelogram: if any pair of fixed points in QC, QD, respectively, are once collinear with P, they will always be so, however the rods are moved about.

NOTE—*This is virtually the same as the omitted* vi. 32. *Ex.* 62, p. 82, *is a particular case of it.*

68. Two triangles are similar and similarly situated; if a third triangle can be drawn to circumscribe the inner and be inscribed in the outer, then its area is a mean proportional between the areas of the original triangles.

NOTE—*Take the centre of similarity of the* $\triangle^s$: *see* vi. *Addenda* (3).

69. ABCD is a quadrilateral circumscribing a circle (centre O) and XOY is perpendicular to the bisector of the angle between BA, CD—X being in AB, and Y in CD—show that AX : BX = CY : DY.

NOTE—*Use Exercise* 41.

70. Prove Exercise 113, p. 190, by vi. *Addenda* (24) and the last Exercise.

NOTE—*This is Newton's original mode of proof.*

71. ABCD is *any* quadrilateral; and M, N are the respective mid points of AC, BD: if MN produced meets AB, BC, CD, DA in P, Q, R, S respectively, then—

$$PA : PB = QC : QB = RC : RD = SA : SD;$$

$$\text{and} \quad MP : MQ = MR : MS = NP : NS = NR : NQ.$$

And if the quadrilateral can have a circle (centre O) inscribed in it, then also—

$$OP : OR = AB : CD;$$

$$\text{and} \quad OQ : OS = BC : DA.$$

72. The following group * are developments of *Note* (4) p. 253—

(1) If we take two maps of the same country, on different scales, and throw one on the other, there will be one spot (and only one) whose position on the one map will be exactly over its position on the other; provided that the contour of the lesser map is wholly within that of the larger.

Also, if the maps are not superposed, but simply laid at random on a table, there will be one point on the table which will represent the same place to whichever map it may be considered to belong.

(2) If corresponding points are taken, one on each map; then the Locus of the point of intersection of corresponding lines through them is a circle.

(3) If a series of parallels is taken on one map; then the Locus of the intersection of each with the corresponding line in the other map, is a straight line.

(4) If a series of concentric circles is taken in one map; then the Locus of the intersection of each with the corresponding circle in the other map is a circle.

NOTE—*Use* vi. *Addenda* (15).

(5) Two corresponding points, one on each map, are held fixed, while the maps are moved about; find the Locus of the centre of similarity.

(6) A pin is put through both maps at a given point; find the Locus of the centre of similarity, as one or other map is turned round.

(7) Find the Loci of corresponding points, one on each map, whose distance apart is constant.

(8) Find the Envelope (cf. *Note* on Exercise 59) of the joins of the points in (7).

(9) If in (5) any two corresponding points, which are at a given distance apart, are fixed, we get a series of Loci; find their Envelope.

(10) Find the Envelope of the Loci in (2), when the points are at a given distance apart.

(11) If a circular disc is placed anywhere on the maps, it must cover at least *some* corresponding points, provided its centre lies within a certain circle; otherwise it will not cover any corresponding points.

* Of this group (1) and (7) are due to Professor Purser, of Queen's College, Belfast; and the rest to Mr. Alexander Larmor, of Clare College, Cambridge.

GENERAL ADDENDA.

SECTION i—MAXIMA AND MINIMA.

Def. If a straight line, or angle, or area, can vary, subject to given limitations; it is said to be *maximum*, when it has its greatest possible value; and *minimum*, when it has its least possible value.

Some cases of *maxima* and *minima* have already occurred : see iii. 7, 8, 15 ; i. *Addenda* (2); iii. *Addenda* (2); vi. *Addenda* (9). Here follow some Theorems which may be regarded as fundamental.

THEOREM (1)—*The sum of the squares on the two segments into which a given line can be divided, is* minimum, *when the line is bisected.*

For, taking fig. (1) of ii. 9,

$$AX^2 + BX^2 = 2\,AM^2 + 2\,MX^2,$$

and $\therefore$ is *min.* when X is at M.

THEOREM (2)—*The rectangle under the two segments into which a given line can be divided, is* maximum *when the line is bisected.*

Taking same fig., we have,

$$AX \,.\, BX = BM^2 - MX^2,$$

and $\therefore$ is *max.* when X is at M.

Cor. Of all rectangles, of given perimeter, the square has the *maximum* area.

THEOREM (3)—*If the rectangle under two lines is given, the sum of the lines is* minimum *when they are equal.*

Follows at once from ii. *Addenda* (3).

Cor. Of all rectangles, of given area, the square has the *minimum* perimeter.

THEOREM (4)—*If the sum of the squares on two lines is given, the sum of the lines is* **maximum** *when they are equal.*

Follows at once from ii. *Addenda* (6).

Note—When any two magnitudes whatever are *commensurable* (so that they can be expressed by x and y units of measurement respectively), Theorems, analogous to the foregoing, are seen to be true, from the two algebraic identities—

$$x^2 + y^2 \equiv 2\left\{\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2\right\};$$

$$xy \equiv \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2.$$

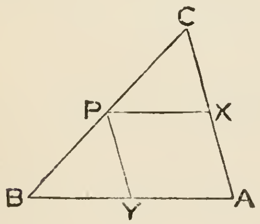
And similar theorems will follow, for the reciprocals of the magnitudes, from the identities—

$$\frac{1}{x^2} + \frac{1}{y^2} \equiv \frac{2\left\{\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2\right\}}{\left\{\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2\right\}^2};$$

$$\frac{1}{x} + \frac{1}{y} \equiv \frac{x+y}{\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2}.$$

Def. When two magnitudes are so related that they vanish together, and that equal increments of the one involve equal increments of the other, the magnitudes are said to **vary one as the other**: such magnitudes will be *maxima* together, and *minima* together.

THEOREM (5)—*The* **maximum** *parallelogram which can be inscribed in a triangle, by drawing parallels to two of its sides, is that formed by drawing the parallels from the mid point of the third side; and its area is half that of the triangle.*



Let PX, PY be $\|^s$ to sides AB, AC, of $\triangle$ ABC, drawn from pt. P in BC.

Then since $\hat{A}$ of ▭ AXPY is const.

$\therefore$ its area varies as AX . AY.

But AX : BP = AC : BC, a const. ratio.

$\therefore$ AX varies as BP.

Simrly. AY „ CP.

$\therefore$ AX . AY „ BP . CP;

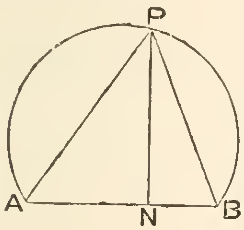
and this, by Theor. (2) is *max.* when P is mid pt. BC.

In that case X, Y are also mid pts. of AC, AB;

and $\therefore$ ▭ AP = $\frac{1}{2}$ △ ABC.

Note—The preceding Theorem is practically equivalent to the omitted vi. 27.

THEOREM (6)—*The* maximum *triangle which can be inscribed in a given segment of a circle, is that formed by joining the mid point of its arc to the extremities of its chord.*



For if APB is given segt., then area △ formed by joining any pt. P on its arc to A, B,

= $\frac{1}{2}$ AB . PN,

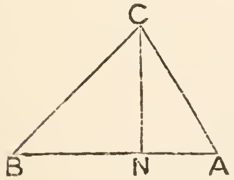
where PN is ⊥ from P on AB.

$\therefore$ area △ APB is *max.* when PN is *max.*

i. e. when P is mid pt. of arc of segt.

Cor. If the base and vertical angle of a triangle are given, the triangle is *maximum* when it is isosceles.

THEOREM (7)—*When two sides of a triangle are given in length, the area of the triangle is* maximum *when they are placed at right angles.*



For let AB, AC be given sides of △ ABC. Draw CN ⊥ to AB.

Then

area △ ABC = $\frac{1}{2}$ AB . CN;

$\therefore$ area $\triangle$ ABC is *max.* when CN is *max.*

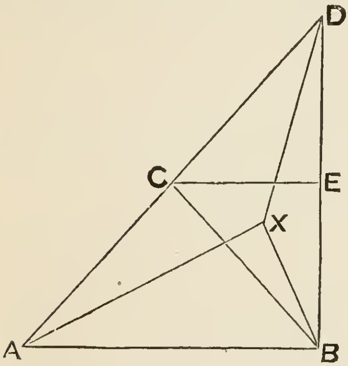
But CN $<$ CA, since $\widehat{CNA}$ is right,

unless CN coincides with CA.

$\therefore$ area is *max.* when $\widehat{CAB}$ is right.

Def. Figures whose perimeters are equal are called **isoperimetrical.**

THEOREM (8)—*Of all isosperimetrical triangles, on the same base, that one of* **maximum** *area is the isosceles.*



Let AB be a fixed base, ABC an isos. $\triangle$ on it, and AXB another $\triangle$, such that

$$AX + BX = AC + CB.$$

Produce AC to D, so that CD $=$ CA.

Draw CE $\parallel$ to AB, to meet DB in E.

Join DX.

Then it is clear that $\triangle$ CED $\equiv$ $\triangle$ CEB.

$\therefore$ AC + CB = AC + CD, and $\therefore$ $<$ AX + DX.

i. e. AX + BX $<$ AX + DX;

$\therefore$ BX $<$ DX;

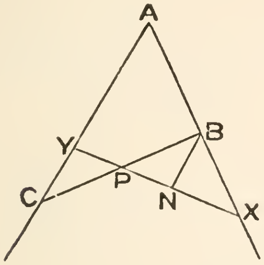
$\therefore$ X lies on same side of CE as AB;

i. e. alt. of $\triangle$ AXB $<$ alt. of $\triangle$ ACB;

$\therefore$ area $\triangle$ AXB $<$ area $\triangle$ ACB.

i. e. $\triangle$ ACB is *max.* under given condns.

THEOREM (9).—*Of all lines, passing through a fixed point, that which determines with two fixed lines the triangle of* minimum *area is the one whose segment, intercepted between the lines, is bisected at the point.*



Let P be the fixed pt., AB, AC the fixed lines.

Take any line XPY, meeting AB in X, and AC in Y.

If PX, PY are not equal, in PX, the greater, take N so that

PN = PY.

Draw NB ∥ to AC, and meeting AB in B.

Join BP, and produce it to meet AC in C.

In $\triangle$s BPN, CPY, since

$$\left.\begin{aligned} P\hat{B}N &= P\hat{C}Y, \\ B\hat{P}N &= C\hat{P}Y, \\ \text{and}\quad PN &= PY; \end{aligned}\right\}$$

$$\therefore\ PB = PC,$$

And $\triangle$ BPN = $\triangle$ CPY.

$\therefore$ $\triangle$ ABC < $\triangle$ AXY, by $\triangle$ BXN.

i. e. for the *min.* $\triangle$ ABC, PB = PC.

Note—The *min.* $\triangle$ is easily constructed by drawing a ∥ from P to AC, meeting AB in E: then EB = EA.

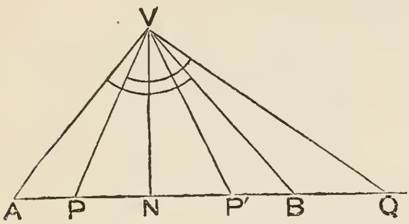
Def. If from a point A a perpendicular AN is drawn to a line XY, and produced to A′, so that NA′ is equal to NA, then A′ is termed the *image* of A, with respect to XY; and if two figures are so situated, on opposite sides of XY, that every point on one is the image of a point on the other, then the figures are said to be *images* of each other, with respect to XY.

Note—The term *reflexion* is sometimes used instead of *image.*

Note—If the plane, in which a figure is situated, is supposed to be turned about the line of reference XY, as a hinge, until it coincides with its original position, the figure will then coincide with its image. Cf. i. 5.

Hence figures which are images of each other are identically equal.

THEOREM (10)—*If the vertical angle and altitude of a triangle are given, its area is* minimum *when it is isosceles.*



Let AVB be an isos. Δ, whose alt. VN and vert. $\widehat{AVB}$ are given.

Take any other Δ PVQ, of same alt. and such that $\widehat{PVQ} = \widehat{AVB}$.

Let Δ VNP′ be the image of Δ VNP, with respect to VN.

$$\text{Then}\quad \widehat{QVB} = \widehat{AVP} = \widehat{BVP'}.$$

$$\therefore\quad QV : VP' = QB : BP' = QB : AP.$$

$$\text{But}\quad QV > VP', \quad \text{i. } \textit{Addenda } (2).$$

$$\therefore\quad QB > AP.$$

$$\therefore\quad PQ > AB.$$

$$\therefore\quad \Delta\, PVQ > \Delta\, AVB.$$

i.e. AVB is the *min.* Δ under the given condns.

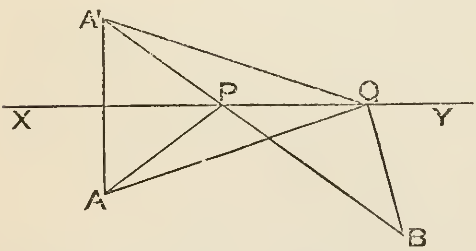
THEOREM (11)—*If* A, B *are two fixed points, and* XY *a fixed line: then, for that point* P *in* XY *at which* AP, BP *make equal angles with* XY—

(α) *when* A, B *are on same side of* XY,

$$AP + BP \text{ is minimum};$$

(β) *when* A, B *are on opposite sides of* XY,

$$AP \sim BP \text{ is maximum}.$$



(α) Let A′ be the image of A, with respect to XY.

Then A′B will cut XY in P,

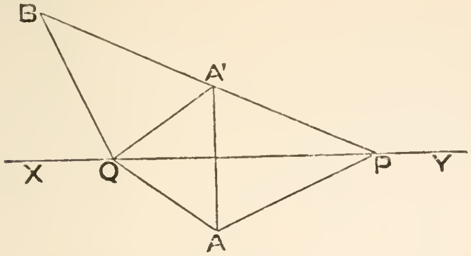
$$\because\quad \widehat{APX} = \widehat{A'PX} = \widehat{BPY}.$$

Let Q be any other pt. in XY; and join QA, QB, QA′.

Then, by the nature of images,

$$AQ + BQ = A'Q + BQ > A'B > AP + BP.$$

i.e. AP + BP is *min.*



(β) Constructing similarly, we have

$$QA \sim QB = QA' \sim QB < BA' < BP \sim PA' < BP \sim AP.$$

i. e. $AP \sim BP$ is *max.*

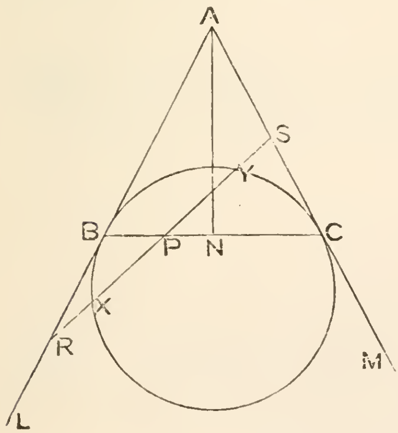
Cor. If B' is the image of B in both figs.,

then (α) $(AP + BP)^2 = AB^2 + AA' \,.\, BB'$,

and (β) $(AP \sim BP)^2 = AB^2 - AA' \,.\, BB'$.

These results can easily be deduced by applying Ptolemy's Theorem to the cyclic quad. $AA'\,BB'$.

THEOREM (12)—*Of all lines which can be drawn through a fixed point, within a fixed angle, that which makes equal angles with the lines forming the angle has the rectangle under its segments* minimum.



Let P be pt. within $\widehat{LAM}$; AN the bisector of $\widehat{A}$; and $PN \perp$ to AN, meeting AL in B and AM in C.

Clearly $AB = AC$; and a ⊙ can be drawn thro. B and C, touching AL, AM at B, C.

Then if any other line is drawn thro. P, meeting AL, AM in R, S, the pts. X, Y, in which it cuts the ⊙, will lie within $\widehat{LAM}$.

$$\text{But} \quad BP \,.\, CP = XP \,.\, YP,$$

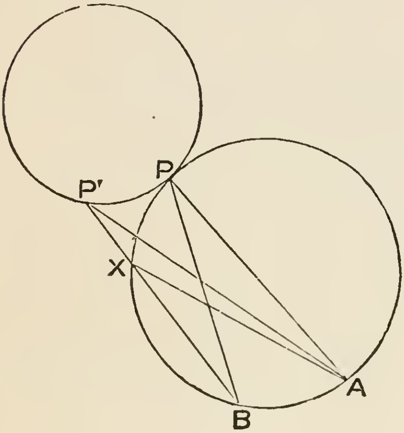
$$\text{and } \therefore \; < RP \,.\, SP;$$

$\therefore$ BPC, making equal $\wedge^s$ with AL, AM, has $BP \,.\, CP$ *min.*

THEOREM (13)—*If* A, B *are fixed points, outside a fixed circle, and* P *a variable point on the circle; then angle* APB *is*—

1°, maximum, *when circle through* A, P, B *has external contact at* P *with fixed circle; and*

2°, minimum, *when internal contact.*



1°, let ⊙ thro. A, B have ext. cont. with fixed ⊙ at P.

Take P′ any other pt. on fixed ⊙.

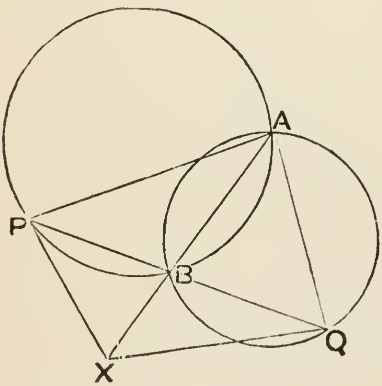
Join P′A, P′B; and let one of them (P′B say) cut ⊙ thro. A, B in X; and join AX.

Then $\widehat{APB} = \widehat{AXB}$, and $\therefore > \widehat{AP'B}$.

i. e. $\widehat{APB}$ is *max.* in this position.

2°, simrly. it can be shown that $\widehat{APB}$ is *min.* when ⊙ APB has internal contact at P.

THEOREM (14)—*Of all lines through one point of intersection of two fixed circles, that one will have the rectangle under the intercepted chords* maximum *which has the tangents to the circles, at its extremities, equal.*



Let AB be common chd. of two intersecting ⊙s; PBQ any line thro. B, terminated by the ⊙s in P Q.

Join PA; and take X, in AB produced, so that $\widehat{BXQ} = \widehat{APB}$.

Then A, P, X, Q are concyclic.

$\therefore$ PB . BQ = AB . BX.

But since $\widehat{BXQ}$ is const., all positions of QX are ||,

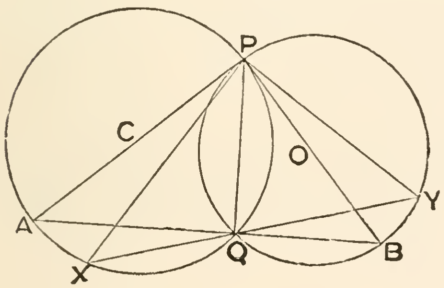
and $\therefore$ BX *max.* when XQ touches ⊙.

And then $\widehat{PAB} = \widehat{PQX} = \widehat{QAB} = \widehat{QPX}$;

so that then XP will touch the other ⊙.

$\therefore$ PB . BQ is *max.* when tangs. at P and Q are equal.

THEOREM (15)—*Of all lines through either point of intersection of two fixed circles, and terminated by the circles, that which forms the* **maximum** *triangle, when its extremities are joined to the other point of intersection, is the one perpendicular to the common chord.*



Let P, Q be pts. of intersec. of ⊙s whose centres are C, O; AQB the line thro. Q ⊥ to PQ; and XQY any other line; so that A, B, X, Y are on the ⊙s.

Since $\widehat{PXQ} = \widehat{PAQ}$, in same segt.

and $\widehat{PYQ} = \widehat{PBQ}$, „

$\therefore$ Δ^s PXY, PAB are simr.

$\therefore$ area Δ PAB : area Δ PXY = $PA^2 : PX^2$.

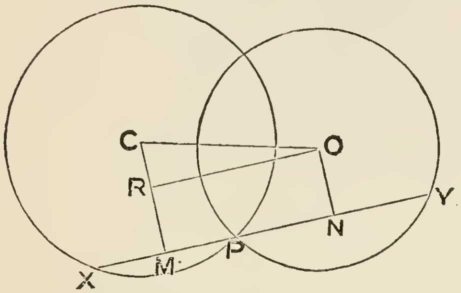
But, since $\widehat{AQP}$ is right, PA is a diam.

$\therefore$ PA > PX.

$\therefore$ area Δ PAB > area Δ PXY.

i.e. Δ PAB is *max.*

THEOREM (16)—*Of all lines through a point of intersection of two circles, and terminated by the circles, the* maximum *is the one parallel to their line of centres; and is twice the join of their centres.*



Let P be a pt. of intersec. of ⊙s whose centres are C, O; XPY any line thro. P, so that X, Y are on the ⊙s.

Draw CM, ON ⊥ to XY; and OR ⊥ to CM.

Then XY = 2 PM + 2 PN = 2 RO,

which < 2 CO, ∵ $\widehat{CRO}$ is right,

unless RO coincides with CO.

∴ XY is *max.* when it is ∥ to CO;

and then it is 2 CO.

Note—If we call such lines as XPY *double chords* of the circles, then the Theorem is—*The* maximum *double chord of two intersecting circles is perpendicular to their common chord.*

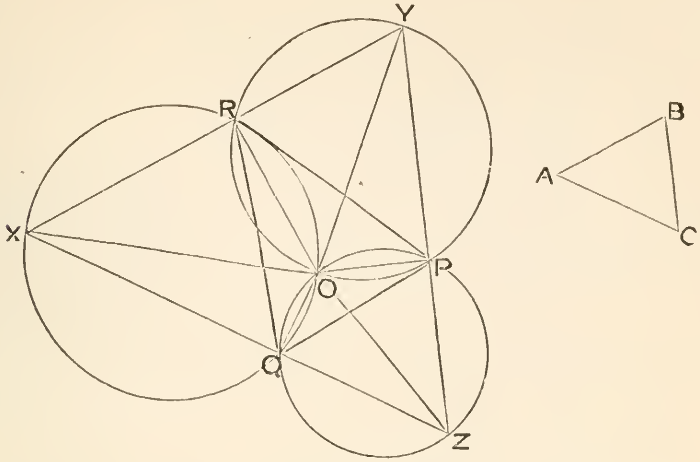
THEOREM (17)—*If* P, Q, R *are given points* (*not in a line*) *and* ABC *a triangle of given species; and if on sides* QR, RP, PQ, *remote from triangle* PQR, *segments are described containing angles* A, B, C *respectively*—

1°, *the circles, being completed, will go through a fixed point* O:

2°, *if any line* YPZ *is drawn so that* Y *is on circle through* P, R; *and* Z *on circle through* P, Q; *and if* YR, ZQ *meet in* X; *then will* X *be on circle through* Q, R:

3°, *the angles* XOY, YOZ, ZOX *are fixed:*

4°, *if* YPZ *is drawn perpendicular to* PO, *then will* ZX, XY *be respectively perpendicular to* QO, RO; *and, in that case,* XYZ *is the* maximum *triangle, of same species as* ABC, *whose sides go through* P, Q, R.



For 1° and 2°; let ⊙s on PQ, PR meet again in O. Join PO, QO, RO.

$$\text{Then } \widehat{POQ} = \text{supplt. } \widehat{PZQ}; \text{ and } \widehat{POR} = \text{supplt. } \widehat{PYR}.$$

$$\therefore \widehat{POQ} + \widehat{POR} + \widehat{PZQ} + \widehat{PYR} = 4 \text{ rt. } \wedge^s = \widehat{POQ} + \widehat{POR} + \widehat{QOR}.$$

$$\therefore \widehat{QOR} = \widehat{PZQ} + \widehat{PYR} = \text{supplt. } \widehat{QXR}.$$

∴ O and X are concyclic with Q, R.

For 3°; $\widehat{XOY} = \widehat{PZQ} + \widehat{QXO} + \widehat{PYO} = \widehat{PZQ} + \widehat{PRQ}$;

and ∴ is fixed.

Simrly. $\widehat{YOZ}$ and $\widehat{ZOX}$ are fixed.

For 4°; $\widehat{ZQO}$ = supplt. $\widehat{ZPO}$; and ∴ is right.

Simrly. $\widehat{YRO}$ is right.

∴, by Theorem (16) Note, △ XYZ has each of its sides *max.*; and ∴, as its species is fixed, it is the *max.* △ under the given condns.

Note (1)—If △ XYZ is fixed, and △ PQR of given species, similar results will hold; and △ PQR will be *min.* when OP, &c., are ⊥ to sides of XYZ.

Note (2)—If △ XYZ is fixed, and RPQ a transversal such that PQ : PR is constant; then the pt. O which (by Ex. 100. p. 188) is common to ⊙s round △s XYZ, XQR, YPR, ZPQ, is fixed.

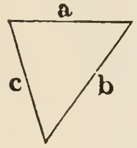
For △s QOR, ZOY are simr., and ∴ OY : OZ = OR : OQ.

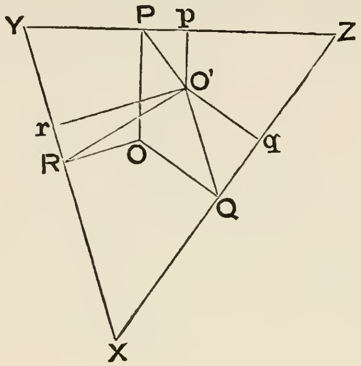
But, since ∧s POQ, POR are fixed, and PQ : PR const., it easily follows that

OR : OQ is const.

∴ O, being on two fixed ⊙s, is fixed.

THEOREM (18)—*If, in the last Theorem, the sides* a, b, c, *of triangle* ABC, *are given; then* O, *the point common to the three circles, is the position for which* a . OP + b OQ + c . OR *is* minimum.





Let sides of $\triangle$ XYZ be $\perp$ to OP, OQ, OR, as in last Theorem.

Take any other pt. O'; and drop O'p, O'q, O'r $\perp^s$ on sides of $\triangle$ XYZ.

Then YZ . OP + ZX . OQ + XY . OR,
= 2 area $\triangle$ XYZ,
= YZ . O'p + ZX . O'q + XY . O'r.

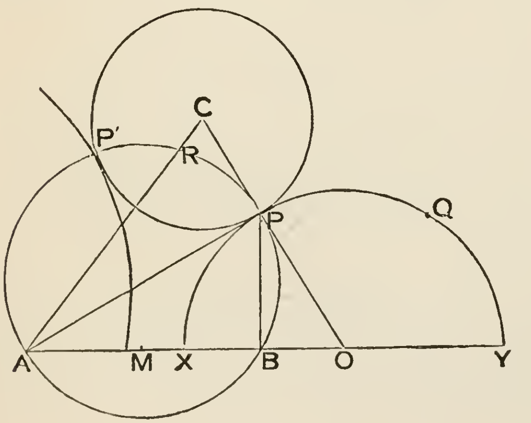
But YZ, ZX, XY are respectively propl. to a, b, c.

$\therefore$ a . OP + b . OQ + c . OR = a . O'p + b . O'q + c . O'r,

and $\therefore$ < a . O'P + b . O'Q + c . O'R.

i.e. is *min.*

THEOREM (19)—*If* A, B *are fixed points, and* P *a variable point on a fixed circle; then the ratio of* PA *to* PB *is* maximum *or* minimum *when* P *is one of the points in which the fixed circle is cut orthogonally by a circle through* A, B. (*T. S. Davies: Philosophical Transactions*, 1836)



Take any pt. X in AB, and produce AB (or BA) to Y, so that Y is the harmonic conjugate (see Section vii) of X with respect to A, B.

Then if Q is any pt. on the ⊙ whose diam. is XY,

$$QA : QB = XA : XB = YA : YB.$$

But if M is mid. pt. of AB, $MX \cdot MY = MA^2$ always. (*Cf.* p. 368)

$\therefore$ X, Y move in opposite directions.

$\therefore$ QA : QB is *max.* when XY *min.*

and ,, *min.* ,, *max.*

Now of all the ⊙s on XY as diam. *two* will touch the given fixed ⊙.

$\therefore$, as in this case Q is to be on the fixed ⊙,

QA : QB, i.e. PA : PB,

is *max.* for the one of these two which touches fixed ⊙ further in direction AB;
and is *min.* ,, ,, ,, ,, ,, ,, BA.

Let C be centre of fixed ⊙. Join CA; and divide it in R, so that

$$CA \cdot CR = CP^2.$$

Then ⊙ thro. A, B, R will cut fixed ⊙ orthogonally—say in P, P′; of which P is further in direction AB.

Let CP meet AB produced in O. With centre O, and radius OP, describe ⊙ cutting AB internally in X, externally in Y.

Then this ⊙ touches fixed ⊙ in P; and X, Y are harmonic conjugates to A, B.

$\therefore$ PA : PB is *max.*

Simrly. P′A : P′B is *min.*

Note (1)—Of course when PA : PB is *max.*; PB : PA is *min.*; and *vice versâ*; so that each pt. P, P′ determines a *max.* and *min.* ratio, according as A is taken in the antecedent or consequent.

Note (2)—When the given ⊙ cuts (as in the fig. it would) the ⊥ bisector of AB, the two ⊙s of contact are on opposite sides of this ⊥, and both contacts are external; otherwise both ⊙s of contact lie on the same side of this ⊥, and the contacts are one external, and one internal.

Note (3)—If a line DE is given (instead of the ⊙ C) then the same reasoning will show that the pts. P, P′, in DE, for which PA : PB is *max.* or *min.*, are given by describing a ⊙ thro. A, B, to cut DE orthogonally: the centre of this ⊙ is, of course, the pt. where the ⊥ to AB, at its mid pt., cuts DE.

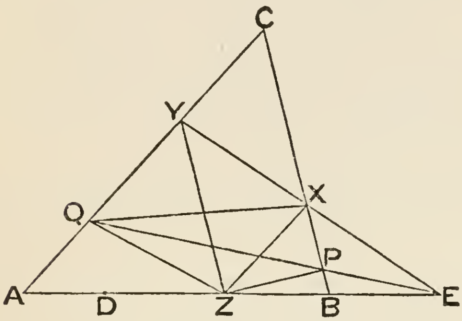
Noticeable cases are, if the given line DE—

1°, goes thro. the mid pt. of AB, when $\widehat{APB}$ is rt.;

or, 2°, goes thro. A, when $\widehat{ABP}$ is rt.;

or, 3°, is ∥ to AB, when $\widehat{PAB} \sim \widehat{PBA}$ is rt.

THEOREM (20)—ABC *is a fixed triangle,* D *a fixed point in* AB, *and* E *in* AB *produced: of all triangles so inscribed in* ABC, *that* D *is their vertex, and that their bases produce through* E, *the* maximum *is that which is so situated that the parallels to* CA, CB *at the extremities of its base, meet on* AB.*



Let EXY cut CB, CA in X, Y, so that XZ, YZ, being ||s to CA, CB, Z is in AB.

Draw any other line EPQ to meet CB, CA in P, Q.

Join PZ, QZ, QX.

Then $\triangle$ PQD : $\triangle$ PQZ = $\perp$ from D on EQ : $\perp$ from Z on EQ.
= ED : EZ, a fixed ratio.

$\therefore$ $\triangle$ PQD is *max.* when $\triangle$ PQZ is *max.*

Now, $\because$ PX is || to ZY, X is farther from QZ than P is.

$\therefore$ $\triangle$ XQZ > $\triangle$ PQZ.

And, $\because$ QY is || to ZX, $\triangle$ XQZ = $\triangle$ XYZ.

$\therefore$ $\triangle$ XYZ > $\triangle$ PQZ.

i. e. $\triangle$ XYZ is *max.* under the given condns.

$\therefore$ so also is $\triangle$ XYD.

Note—By drawing the fig. it will be found that the proof is exactly the same when PQ is taken on the other side of XY.

Def. If four points (no three of which are in the same line) are joined by four lines, in such a manner that two of the joins cross each other, the resulting figure is called a **cross quadrilateral**: the four joins are called the *sides* of the quadrilateral; the four points are called its *corners*; and if the remaining pair of joins are drawn, they are called its *diagonals*.

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* *Mulcahy's Modern Geometry*, p. 88.

Note (1)—The appropriateness of these terms will become evident by considering that, if an ordinary quadrilateral is imagined to consist of four bars freely jointed at its corners, then by twisting the triangle formed by two adjacent bars and the line joining their extremities, round the join as axis, until it returns to the original plane of the quadrilateral, a cross quadrilateral is formed.

Note (2)—A cross quadrilateral obviously consists of two triangles of which a pair of angles are vertically opposite.

Note (3)—If **a**, **b**, **c**, **d** denote the successive sides, and **x**, **y** the diagonals of a quadrilateral; then—

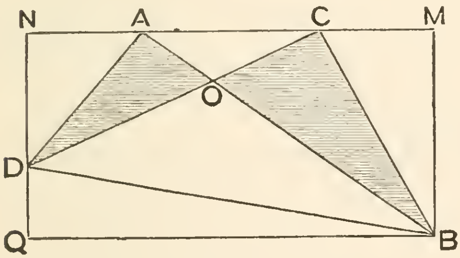
1°, when it is *not* crossed, $xy < ac + bd$;

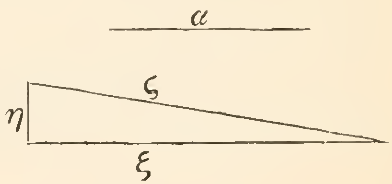
2°, but when it *is* crossed, $xy > ac \sim bd$—

unless the four corners are concyclic, when both inequalities become equalities.

Cf. vi. *Addenda* (9).

THEOREM (21)—*If the lengths of the four sides of a crossed quadrilateral are given, the difference of the areas of the triangles composing it is* minimum *when its corners are concyclic.*





Let ABCD be a crossed quad. whose sides AB, CD cross at O.

Draw BM, DN ⊥ to AC; and BQ ⊥ to ND.

Let **a**, **b**, **c**, **d** respectively denote the fixed lengths AB, BC, CD, DA; and **x**, **y** the variable lengths AC, BD.

$$\text{Then}\quad a^2 - b^2 = x\,(AM + CM),$$

$$\text{and}\quad c^2 - d^2 = x\,(CN + AN);$$

$$\therefore\quad a^2 - b^2 + c^2 - d^2 = 2x \,.\, MN = 2x \,.\, BQ.$$

$$\therefore\quad x \,.\, BQ \text{ is fixed.}$$

Also

$$\triangle BOC \sim \triangle DOA = \triangle BAC \sim \triangle DAC = \tfrac{1}{2}x\,(BM \sim DN) = \tfrac{1}{2}x \,.\, DQ.$$

$$\therefore\quad x \,.\, DQ = 2 \text{ diff. of areas in question.}$$

Now take any fixed length a; and let ξ, η, ζ be lengths such that rects. $a\xi$, $a\eta$, $a\zeta$ are respectively equal to rects. $\mathbf{x} \cdot \mathsf{BQ}$, $\mathbf{x} \cdot \mathsf{DQ}$, $\mathbf{x} \cdot \mathsf{BD}$.

Then $\xi : \mathsf{BQ} = \eta : \mathsf{DQ} = \zeta : \mathsf{BD}$.

$\therefore$ ξ, η, ζ will form a $\triangle$ simr. to $\triangle$ BQD.

$\therefore \quad \eta^2 + \xi^2 = \zeta^2$.

But ξ is fixed, $\because \; a\xi = \mathbf{x} \cdot \mathsf{BQ}$.

$\therefore$ η is *min.* when ζ is *min.*

$\therefore$ $a\eta$ (i.e. $\mathbf{x} \cdot \mathsf{DQ}$) is *min.* when $a\zeta$ (i.e. $\mathbf{xy}$) is *min.*

But $\mathbf{xy} > \mathbf{ac} \sim \mathbf{bd}$, unless A, B, C, D are concyclic.

$\therefore$ $\mathbf{x} \cdot \mathsf{DQ}$ is *min.* when quad. is cyclic,

and then so also is diff. of areas AOD, COB.*

Note—If the quad. is an ordinary one, *not* crossed, the same reasoning proves that the *sum* of the $\triangle^s$, i. e. the area of the quad., is *maximum* when it is cyclic.

Cor. Given the sides $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, of a quad., the acute angle between its diags. is—

1°, *max.* when quad. is cyclic but not crossed;

2°, *min.* ,, ,, and crossed—

except when $\mathbf{a}^2 - \mathbf{b}^2 + \mathbf{c}^2 - \mathbf{d}^2 = 0$, and then the diags. are at right angles; and the area is *min.*

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* This proof and Cor. are due to Professor Purser of Queen's College, Belfast, by whom they were communicated to the Editor. Professor Purser originally proposed the Theorem for demonstration in an examination paper in 1880; and afterwards in the *Educational Times*: see the *Reprint*, *Vol. XXXV*, p. 99.

Mr. Young (of Derry) sent the Editor a proof of this Theorem, on more elementary principles than the above: it has since been published in the *Educational Times*: li. p. 88.

EXERCISES ON MAXIMA AND MINIMA.

1. Find the *minimum* distance between two non-intersecting circles.

2. The *maximum* rectangle which can be inscribed in a given circle is a square.

NOTE—*Consider one of the* Δ^s *formed by the diags., and apply Theorem* (6).

3. Find the *maximum* rectangle which can be inscribed in a given semi-circle.

NOTE—*Draw the radii to the two corners on the arc; and use Theorem* (7).

4. The area of the triangle which a variable tangent to a fixed circle makes with two fixed tangents (or these produced) is *maximum* or *minimum* when the variable tangent is bisected at its point of contact.

NOTE—*Use Theorem* (10).

5. If the diagonals of a parallelogram are given, its area is *maximum* when it is a rhombus.

6. If the diagonals of any quadrilateral are given, its area will be *maximum* when they cut at right angles; but is independent of the lengths of the segments they make.

NOTE—*See* i. *Addenda* (28); *and use Theorem* (7).

7. The *minimum* square which can be inscribed in a given square is half its area.

NOTE—*Deduce from Theorem* (1).

8. C is the centre of a given circle, A a point outside it; if AXY is drawn to cut the circle in X, Y, find when area triangle CXY is *maximum*.

9. C is the centre of a fixed circle, XY a variable chord of the circle; if CP is perpendicular to XY, find when

1°, CP + PX is *maximum*.

2°, CP . PX ,,

10. In Theorem (13) if A, B are within the circle, similar results are true.

11. A, B are fixed points within a fixed circle; if P is a variable point on the circle, and PA, PB produced, meet the circle in X, Y; find when XY is *maximum*.

NOTE—*Use the last Exercise.*

12. A is a fixed point on a fixed semi-circle, B the mid point of a fixed radius; if P is a variable point on the circle, find when AP + 2 BP is *minimum* or *maximum*.

NOTE—*Apply* vi. *Addenda* (9) *to quad.* APBO, *where* O *is centre.*

13. If in Theorem (13) a line is substituted for the circle, show that similar results are true.

14. P is a variable point on a fixed circle; A, B two fixed points: find when
1°, $PA^2 + PB^2$ is *maximum* or *minimum.*
2°, PA + PB „ „ (*Alhazen's Problem.*)

NOTE—*For* 1°, *the ends of the diameter bisecting* AB: *for* 2°—*though, by Theorem* (11), *we can see* when *it happens—the points cannot be found with the ruler and compasses only; but a solution on other grounds, by Dr. Curtis, is given in the Educational Times, Vol. XXXIX,* p. 59, *of the Reprint.*

15. Draw the *minimum* tangent from a point, in a given line, to a given circle.

16. Given an angle, and a fixed point on its bisector, show that the line through the point, which makes equal angles with the arms of the angle, is the *minimum* line, and cuts off the *minimum* triangle.

17. Given an angle, show that of all lines which can be drawn across it to form a triangle of given area, the *minimum* is that which makes equal angles with the arms of the angle.

18. Inscribe the *maximum* triangle in a given circle.

19. Find P, in the diameter AB of a circle, so that PQ being perpendicular to AB, and meeting the circle in Q, AP . PQ is *maximum.*

20. If TA, TB are fixed tangents to a fixed circle, and P a variable point on the circle, find when the rectangle under, 1°, the perpendiculars from P on TA, TB, 2°, the perpendiculars from A, B on the tangent at P, are respectively *maxima* and *minima.*

NOTE—*Use* vi. *Addenda* (13).

21. Given a circle and two lines at right angles, not cutting it, find the points on the circle the sum of whose distances from the lines is *maximum* or *minimum.*

22. From a fixed point O on the production of a diameter of a circle, draw a secant OPQ, so that the difference of the perpendiculars from P, Q on that diameter may be *maximum.*

NOTE—*Use Theorem* (20).

23. O is a fixed point in the production of side CB, of fixed triangle ABC: draw OXY to meet AB, AC in X, Y, so that the difference of perpendiculars from X, Y on BC may be *maximum.*

NOTE—*Use Theorem* (20).

24. ABC is a fixed triangle, P a variable point within it: find when—1°, $\Sigma\, PA^2$; and, 2°, $\Sigma\, PA$; are *minima.*

NOTE—*For* 1°, *see* ii. *Addenda* (16); *and for* 2°, *on* BC *describe outwardly an equilat.* △BDC; *then, if* AD *cuts* ⊙ *round* BDC *in* P, *it can be easily proved that* $\Sigma\, PA$ *is min.* P *in* 2° *is called* **Fermat's point.**

25. AB is the diameter of a semi-circle, and P a variable point on its arc; find when $a \cdot PA + b \cdot PB$ is *maximum*, if a, b are given whole numbers.

26. A and B are fixed points, one inside and one outside a fixed circle (centre C): if BC, AC are respectively m and n times the radius, show that the points in which the circle through A, B, C cuts the fixed circle, determine the positions of P, Q, on the fixed circle, for which—

1°, $m \cdot PA + n \cdot PB$ is *minimum*; and,

2°, $m \cdot QA \sim n \cdot QB$ is *maximum*—

C, P being on opposite, and C, Q on the same side of AB.

27. Find P in the bisector of angle A, of triangle ABC, so that the difference of the angles PBC, PCB is *maximum*; and show that then the sum of these angles is half the angle BAC.

NOTE—*If* $AC < AB$, *let* $\perp$ *from* C *on bisector meet* AB *in* D; *draw* $\odot$ *thro.* B, D *to touch bisector in* T: *then* $\odot$ *thro.* B, T, C *will cut bisector in* P.

28. If the number of sides of a polygon is fixed, and its corners lie on fixed lines, show that when its perimeter is *minimum* the fixed lines bisect its angles externally.

NOTE—*Use Theorem* (11) (a).

29. Of all isoperimetrical polygons, of a given number of sides, that of *maximum* area is the equilateral.

30. If all the sides of a polygon, excepting one, are given in length, show that its area is *maximum* when the remaining side is the diameter of a semicircle, whose arc goes through all the corners.

Deduce from this that if all the sides of a polygon are given in length its area is *maximum* when its corners are concyclic.

31. Given all the sides but one of a polygon, and that the two sides adjacent to the unknown side are parallel; find when the area of the polygon is *maximum*.

32. (1) Of all polygons of a given number of sides, inscribed in a given circle, show that the regular one has its area and perimeter *maximum*.

(2) Of all polygons of a given number of sides, described about a given circle, show that the regular one has its area and perimeter *minimum*.

33. One side of a quadrilateral is fixed; the opposite side goes through a fixed point; the variable corners move on fixed lines: find when the area is *maximum*.

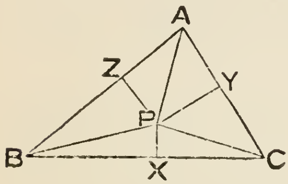
NOTE—*Can be reduced to Theorem* (20).

SECTION ii—CONCURRENCY AND COLLINEARITY.

THEOREM (1)—*If* X, Y, Z *are points in the sides* BC, CA, AB of a triangle ABC, *such that the perpendiculars to the sides at these points are concurrent, then*—

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0;$$

and conversely.



Let P be the pt. of concurrence.

Then joining P with A, B, C, we have

$$\begin{aligned}&(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2),\\ &= BP^2 - CP^2 + CP^2 - AP^2 + AP^2 - BP^2,\\ &= 0.\end{aligned}$$

For the converse; let $\perp^s$ at X, Y meet in P.

Suppose $PZ' \perp$ to AB. Then, by preceding part,

$$(BX^2 - CX^2) + (CY^2 - AY^2) + (AZ'^2 - BZ'^2) = 0;$$

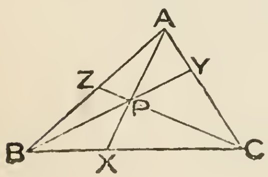
and $\therefore = (BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2)$, by hypothesis.

$\therefore$ Z and Z′ are the same point.

THEOREM (2)—*(Ceva's) When three lines* AX, BY, CZ, *drawn from the corners* A, B, C *of a triangle* ABC, *to meet its opposite sides in* X, Y, Z, *are concurrent, then*—

$$(AZ : ZB)\ (BX : XC)\ (CY : YA) = 1;$$

and conversely.



Let P be the point of Concurrence.

Then by vi. *Addenda* (20),

$$AZ : ZB = \triangle APC : \triangle BPC,$$
$$BX : XC = \triangle BPA : \triangle CPA,$$
$$CY : YA = \triangle CPB : \triangle APB;$$

$\therefore$, compounding the ratios, and recollecting that the ratio compounded of reciprocal ratios is unity, we get

$$(AZ : ZB)\ (BX : XC)\ (CY : YA) = 1.$$

For the converse; let AX, BY cut in P; and suppose that CP meets AB in Z′: then, by preceding part,

$$(AZ' : Z'B)\ (BX : XC)\ (CY : YA) = 1;$$

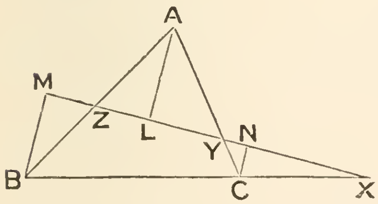
and $\therefore$ $= (AZ : ZB)\ (BX : XC)\ (CY : YA)$, by hypothesis.

$\therefore$ Z and Z′ are the same point.

THEOREM (3)—*(Menelaus') If three points* X, Y, Z, *lying respectively on the three sides* BC, CA, AB *of a triangle* ABC, *are collinear, then—*

$$(AZ : ZB)\ (BX : XC)\ (CY : YA) = 1;$$

and conversely.



From A, B, C let $\perp^s$ AL, BM, CN be drawn respectively on XYZ.

Then by simr. $\triangle^s$, $AZ : ZB = AL : BM,$

$$BX : XC = BM : CN,$$

and $CY : YA = CN : AL.$

$$\therefore\ (AZ : ZB)\ (BX : XC)\ (CY : YA) = 1.$$

For the converse; let XY meet AB in Z′: then, by preceding part,

$$(AZ' : Z'B)\ (BX : XC)\ (CY : YA) = 1;$$

and $\therefore$ $= (AZ : ZB)\ (BX : XC)\ (CY : YA)$, by hypothesis.

$\therefore$ Z and Z′ are the same point.

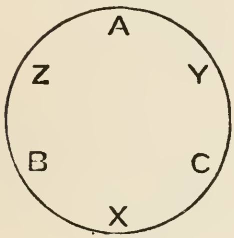
Def. Any line drawn across a system of lines is called a *transversal* of that system.

Hence the preceding Theorem may be expressed thus—*If a transversal cuts the sides (or sides produced) opposite corners* A, B, C, *of triangle* ABC, *in* X, Y, Z, *respectively, then—*

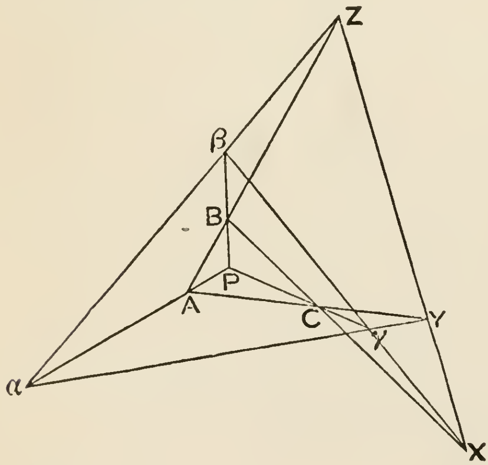
$$(AZ : ZB)\ (BX : XC)\ (CY : YA) = 1;$$

and conversely, if this relation holds between the segments into which the sides of a triangle ABC *are cut (externally or internally) in* X, Y, Z, *then* XYZ *is a transversal of the triangle.*

Note—If in the above result, the antecedents of the ratios are exchanged among themselves, or the consequents among themselves, the result is still true; or we may take the reciprocals of all the ratios. In order to recollect the way in which the letters, in the three preceding Theorems, are to be taken, place them round a circle, as in the accompanying figure, where the corners of the $\triangle$ are placed as A, B, C; and the points in the sides respectively opposite to them as X, Y, Z; then, beginning at *any* point, take the successive letters, *going one way round.*



THEOREM (4)—*(Desargues') If two triangles are so placed that their corners connect concurrently, then their corresponding sides intersect collinearly; and conversely.*



Let ABC, $\alpha\beta\gamma$ be Δ^s such that Aα, Bβ, Cγ are concurrent in P.

Let BC, $\beta\gamma$ meet in X;
„ AC, $\alpha\gamma$ „ Y;
„ AB, $\alpha\beta$ „ Z.

Then $\because$ $\triangle$ P$\alpha\beta$ is cut by transversal ZBA,

$\therefore$ $(\alpha Z : Z\beta)\ (\beta B : BP)\ (PA : A\alpha) = 1.$

And $\because$ $\triangle$ $\mathrm{P}\beta\gamma$ is cut by transversal XCB,

$$\therefore \quad (\beta\mathrm{X} : \mathrm{X}\gamma) \; (\gamma\mathrm{C} : \mathrm{CP}) \; (\mathrm{PB} : \mathrm{B}\beta) = 1.$$

Also $\because$ $\triangle$ $\mathrm{P}\alpha\gamma$ is cut by transversal YCA,

$$\therefore \quad (\gamma\mathrm{Y} : \mathrm{Y}\alpha) \; (\alpha\mathrm{A} : \mathrm{AP}) \cdot (\mathrm{PC} : \mathrm{C}\gamma) = 1.$$

$\therefore$, compounding these ratios, and recollecting that compounds of reciprocal ratios are unity, we get

$$(\alpha\mathrm{Z} : \mathrm{Z}\beta) \; (\beta\mathrm{X} : \mathrm{X}\gamma) \; (\gamma\mathrm{Y} : \mathrm{Y}\alpha) = 1.$$

$\therefore$ X, Y, Z are collinear.

For the converse; if X, Y, Z (the respective intersections of BC, $\beta\gamma$, of AC, $\alpha\gamma$, and of AB, $\alpha\beta$) are collinear, let γC, βB meet in P.

Then $\triangle$s $\mathrm{B}\beta\mathrm{Z}$, $\mathrm{C}\gamma\mathrm{Y}$ have joins of corresponding corners concurrent, viz., BC, $\beta\gamma$, ZY, meeting in X.

$\therefore$, by first part, intersections of βB, γC, of ZB, YC, and of Zβ, Yγ are collinear;

i. e. P, A, α are in one line.

Aα, Bβ, Cγ are concurrent.

Def. Two triangles, related as in the preceding Theorem, are said to be **in perspective**; and the point of concurrency, and line of collinearity, are termed respectively, the **centre** and **axis of perspective**.

Note—The appropriateness of the term *perspective* will be readily seen, by any one who has a slight knowledge of perspective drawing, from the following consideration—which also gives a proof of the Theorem.

Let ABC, $\alpha\beta\gamma$ be in different planes, not ||, so that P is the vertex of a pyramid, of which ABC, $\alpha\beta\gamma$ are triangular sections.

Then planes of ABC, $\alpha\beta\gamma$ will intersect in a line (L say).

But AB, $\alpha\beta$, being in one plane, and not ||, must meet.

$\therefore$ their pt. of meeting is in L.

Simrly. BC, $\beta\gamma$ meet in L; and so also do CA, $\gamma\alpha$.

And this remains true if plane of ABC is turned about L as a hinge.

Let it be turned into coincidence with plane of $\alpha\beta\gamma$: then *Desargues' Theorem* follows.

Def. Any number of collinear points, when taken in connection with each other, is termed **a range**.

Def. Any number of concurrent lines, when taken in connection with each other, is termed **a pencil**: the separate lines are termed **rays**; and the point of concurrency is termed the **focus** of the pencil.

Def. Let a finite line AB be divided, internally in X, and externally in Y, then the ratios of the segments are AX : BX and AY : BY; and the ratio of the rectangles got by taking the terms of these ratios cross-ways,

$$\text{thus} \left\{ \begin{matrix} AX & : & BX \\ & \times & \\ AY & : & BY \end{matrix} \right\} \text{viz. } AX \,.\, BY : AY \,.\, BX$$

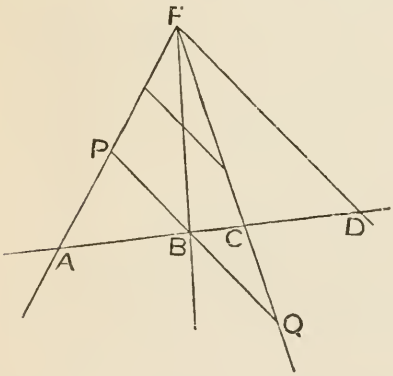
is called a *cross-ratio* (or an *anharmonic-ratio*) of the four segments.

Note—If we write the ratios thus—$\dfrac{AX}{BX}$ and $\dfrac{AY}{BY}$; and *assume* that ratios, expressed in a fractional form, may be manipulated like arithmetic fractions; then, since

$$\frac{\dfrac{AX}{BX}}{\dfrac{AY}{BY}} = \frac{AX \,.\, BY}{AY \,.\, BX},$$

we see that a cross ratio of four segments is the *ratio of the ratios* of the segments; but the assumption implies that the segments are commensurable.

THEOREM (5)—*If a fixed pencil of four rays is cut by a transversal, the cross-ratio of the four segments of the transversal is invariable.*



Let pencil (focus F) be cut by any transversal in A, B, C, D.

Thro. B draw the || to FD, meeting FA in P, and FC in Q.

$$\text{Then} \quad AB : AD = PB : FD,$$

$$\text{and} \quad CD : BC = FD : QB.$$

$\therefore$ $AB \,.\, CD : AD \,.\, BC = PB : QB$, a const. ratio.

$\because$ the cross-ratio $AB \,.\, CD : AD \,.\, BC$ is constant for all transversals.

Note (1)—The cross-ratio of a range A, B, C, D is denoted by (ABCD); and that of a pencil (as above) by F (ABCD).

By *Euler's Theorem* [ii. *Addenda* (8)], viz. that for a range A, B, C, D,

$$AB \, . \, CD + AD \, . \, BC = AC \, . \, BD,$$

we see that, if any one of the cross-ratios

$$AB \, . \, CD : AD \, . \, BC; \quad AB \, . \, CD : AC \, . \, BD; \quad AD \, . \, BC : AC \, . \, BD,$$

or their reciprocals, is constant, so is also each of the others.

Hence, if in either term of a cross-ratio two pairs of letters are interchanged, the ratio remains constant.

Def. If two ranges of points (or pencils of rays) have the same cross-ratio, they are said to be *equi-cross*.*

From Theorem (5) we have—

Cor. (1). If the rays AF, BF, CF, DF are produced, the cross-ratio of the new pencil formed is the same as that of the original pencil.

Cor. (2). If two pencils are equi-cross, and have three rays in common, they have a fourth ray in common.

Cor. (3). If the successive angles of one pencil are respectively equal to the successive angles of another pencil, the pencils are equi-cross.

For the pencils can be superposed so as to coincide.

Cor. (4). If four *fixed* points on a circle are joined to a fifth *variable* point, the pencil has a constant cross-ratio.

For the successive angles formed by the joins are constant.

Note (2)—The cross-ratio of the pencil formed by joining a variable point P on a circle to four fixed points A, B, C, D, on the same circle, is denoted by P (ABCD); and its value is that of the range formed by any transversal to the pencil.

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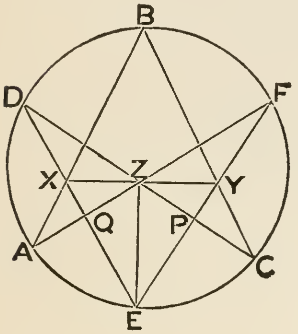
* The term *cross-ratio* was given by the late Professor Clifford in his *Elements of Dynamic*, p. 42. *Equi-cross* was suggested by the late Professor Townsend: see *Taylor's Ancient and Modern Geometry of Conics*, p. 250, Note.

THEOREM (6)—*If* A, B, C, D, E, F *are any six points on a circle, which are joined successively in any order; then the intersections of the first and fourth, of the second and fifth, and of the third and sixth of these joins (produced when necessary) are collinear.*



Let AB, DE cut in X;
" BC, EF " Y;
" CD, FA " Z.
Join ZX, ZY, ZE.
Let ZA cut XE in Q;
and ZC " YE " P.

Then

Z(FYPE) = C(FYPE) = C(FBDE) = A(FBDE) = A(QXDE) = Z(QXDE).

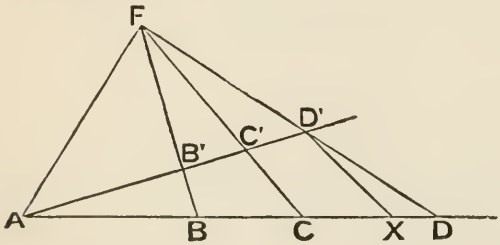
∴, since the pencils are equi-cross and have three rays in common, their fourth ray is in common:

i. e. XZ, ZY are in the same line.

Note (1)—In the single case in which the points are joined in their successive order round the circle, the joins form a hexagon; and the Theorem becomes—*The intersections of the opposite sides of a cyclic hexagon are collinear.* This is known as *Pascal's Theorem.*

Note (2)—Any such line as XZY is called a *Pascal Line*: there are sixty of them, corresponding to the sixty different ways in which the six points can be joined.

THEOREM (7)—*If two ranges of four points, not on the same line, have a common point, and are equi-cross; then the joins of corresponding points are concurrent.*



Let ABCD, AB′C′D′ be two ranges, having A common, and (ABCD) = (AB′C′D′).

Let BB′, CC′ meet in F.

Join FD′; and let it meet ABCD in X. Join FA.

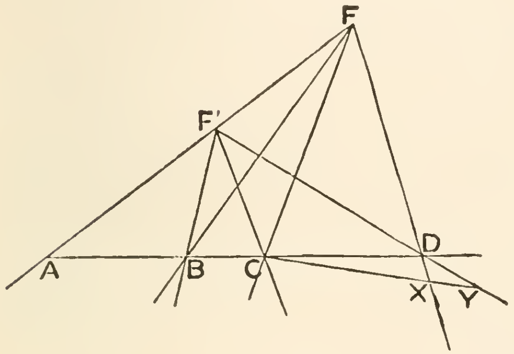
Then F (ABCX) = F (AB′C′D′) = F (ABCD).

∴, by *Theor.* (5) *Cor.* (2), FX and FD are the same ray:

i. e. X and D are the same pt.

∴ BB′, CC′, DD′ are concurrent.

THEOREM (8)—*If two pencils of four rays, not from the same focus, have a common ray, and are equi-cross; then the intersections of corresponding rays are collinear.*



Let F (ABCD) and F′ (ABCD) have a common ray FF′, and the same cross-ratio.

Produce BC both ways, so as to cut FF′ in A, FD in X, and F′D in Y.

Then F (ABCX) = F (ABCD) = F′ (ABCD) = F′ (ABCY) = F (ABCY).

∴, by *Theor.* (5) *Cor.* (2), FX and FY are the same ray:

i. e. X and Y coincide;

and ∴ must coincide with D;

∴ B, C, D are collinear.

Note—The two preceding Theorems may be concisely summed up thus—*If two ranges of points (or pencils of rays)* ABCD, abcd, *are equi-cross; then, if* (A, a) *coalesce, the pairs* (B, b) (C, c), (D, d) *will connect concurrently or intersect collinearly.*

EXERCISES ON CONCURRENCY AND COLLINEARITY.

1. By means of Theorem (1) prove that the following sets of three lines are, in each case, concurrent—

(1) The perpendiculars to the sides of a triangle at their mid points:

(2) the altitudes of a triangle:

(3) The perpendiculars to the sides of a triangle at the points where the sides (not produced) are touched by the three ex-circles:

(4) The two perpendiculars at the points where two ex-circles touch the sides of a triangle, produced through the same corner, and the perpendicular at the point where the in-circle touches the side opposite that corner:

(5) The three tangents at the points of contact of three circles which touch two and two:

(6) Any three perpendiculars to the sides of a triangle, *if* the three perpendiculars respectively equidistant from the mid points of the sides are concurrent:

(7) The three perpendiculars from the corners of one triangle on the sides of another, *if* the three corresponding perpendiculars from the corners of the second on the sides of the first are concurrent.

2. By means of *Ceva's Theorem* prove that the following sets of three lines are, in each case, concurrent—

(1) The medians of a triangle:

(2) The bisectors of the internal angles of a triangle; or of two external and the third internal:

(3) The joins of the points of contact, of the in-circle of a triangle with the opposite corners. (*The point of concurrence is called* **Gergonne's point**)

(4) The altitudes of a triangle:

(5) The joins of the corners of a triangle to three of the points in which the opposite sides are cut by a circle, *if* the joins to the other three points of section are concurrent:

(6) The joins of the corners of a triangle to points on the opposite sides, *if* the joins to the points respectively equidistant from the mid points of the sides are concurrent. (*The points of concurrence are called* **isotomic conjugates**)

(7) The joins of the corners of a triangle to the points in the opposite sides where they are met by lines bisecting the angles (all internally, or two externally and one internally) between three lines drawn from any point to the corners of the triangle.

3. By means of *Menelaus' Theorem* prove that the following sets of three points are, in each case, collinear—

(1) The points in which the three external bisectors of the angles of a triangle meet the opposite sides produced :

(2) The points in which two internal and one external bisector of the angles of a triangle meet the opposite sides (produced if necessary) :

(3) Three points on the sides of a triangle (produced if necessary) *if* the three respectively equidistant from the mid points of the sides are collinear :

(4) The points in which two of the internal and one external bisector of the angles between three lines drawn from any point to the corners of a triangle meet its opposite sides.

4. By means of *Desargues' Theorem* prove that—

(1) When three triangles are in perspective, two and two, and have a common axis of perspective, their three centres of perspective are collinear :

(2) When three triangles are in perspective, two and two, and have a common centre of perspective, their three axes of perspective are concurrent. (*Chasles, Géométrie Supérieure*, pp. 283, 284 : see also *Townsend, Vol. I, p.* 194, where these Theorems are generalised for any rectilineal figure.)

5. Prove the last Exercise, as in the Note to Theorem (4) by considering the triangles not in a plane.

6. Three circles touch the sides of a triangle ABC at the points where the in-circle touches them ; and the circles touch each other in P, Q, R ; show that AP, BQ, CR are concurrent.

7. Lines from two corners of a triangle divide the opposite sides in same ratio ; if the third corner is joined to their intersection, this join produced will either bisect the third side, or divide it in duplicate of the above ratio.

8. ABC is a triangle, O a point within it ; if AO, BO, CO meet BC, CA, AB in X, Y, Z respectively; and YZ, CB, produced meet in P ; then—

$$BX : CX = BP : CP.$$

NOTE—*Combine Ceva's and Menelaus' Theorems.*

9. If a variable transversal cuts the sides of a fixed triangle in X, Y, Z ; and the ratio of XY to YZ is fixed ; then the circum-circle of any one of the three triangles cut off by the transversal goes through a fixed point.

10. If through one of the points of intersection of two circles two lines YXZ, QPR are drawn at right angles ; so that X, P are on the line of centres ; Y, Q on one circumference ; and Z, R on the other ; then—

$$XY : XZ = PQ : PR.$$

NOTE—*Use Menelaus' Theorem.*

11. From the corners B, C, of a triangle ABC, BX, CY are drawn parallel to the opposite sides, so as each to be equal to a given length; if parallels to the adjacent sides, through X, Y, meet in Z; then XC, YB, ZA are concurrent.

12. If from any point within a triangle perpendiculars are dropped on its sides, and the circle through the feet of these perpendiculars cuts the sides again; then the perpendiculars at these last points of section are concurrent.

13. The perpendiculars from the corners of a triangle, on the sides of its pedal triangle, are concurrent at the circum-centre of the original triangle.

14. If twelve perpendiculars are drawn to the sides of a triangle, from the four centres of its circles of contact, these perpendiculars will be concurrent, three by three, in four points which are the centres of the circum-circles of the four ex-central triangles.

NOTE—*The four* **ex-central triangles** *are formed by joining the ex- and in-centres: the* **principal ex-central triangle** *having its corners the ex-centres.*

15. In any triangle, show that the N. P.-circle bisects all lines from the orthocentre to the circumference of the circum-circle; and also bisects all lines from a corner to the circumference of the circle through the orthocentre and the other two corners: hence prove that—

If in each ex-central triangle, its orthocentre is joined with its circum-centre, the four joins are concurrent in the centre of the N. P. circle of the four ex-central triangles.

NOTE—*The last five Exercises are proved in Booth's New Geometrical Methods, Vol. II, pp.* 263, 285, 300, 319, 320.

16. If A, B, C are three collinear points, and a, b, c three other collinear points; then the intersections of Ab, Ba, of Bc, Cb, and of Ca, Ac, are collinear.

NOTE—*Use Theorem* (8).

17. If A, B, C are three concurrent lines, and a, b, c three other concurrent lines; then (denoting the intersection of A and a by Aa) the joins of Ab, Ba, of Bc, Cb, and of Ca, Ac are concurrent.

(*Chasles, Géométrie Supérieure*, p. 294)

18. Four triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$, are so situated that the first is in perspective with the second, the second is in perspective with the third, the third with the fourth, and the fourth with the first; where similarly lettered corners are joined: if the four centres of perspective are collinear, prove that the four axes of perspective are concurrent. (*E. T.* xxix. 53)

SECTION iii—CENTRES OF SIMILITUDE.

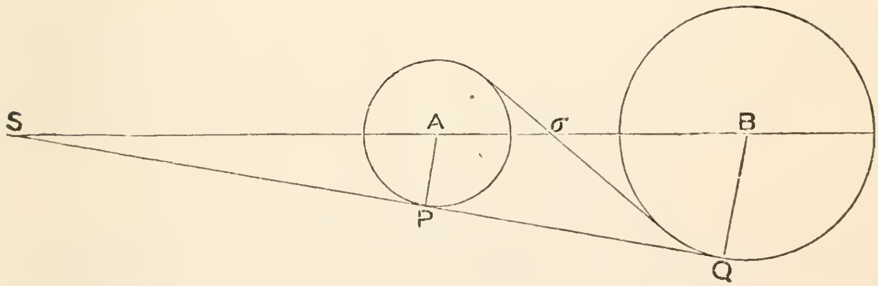
Note—By ⊙ A, it is to be understood that we mean to indicate *the circle whose centre is* A; where, from the construction of the figure, no ambiguity can arise.

Def. If the join of the centres A and B, of two circles, is divided externally in S, and internally in σ, so that

$$SA : SB = \text{radius of} \odot A : \text{radius of} \odot B = \sigma A : \sigma B,$$

the point S is called the *external centre of similitude*, and the point σ is called the *internal centre of similitude* of the two circles.

THEOREM (1)—*A tangent drawn to one of two circles, from either centre of similitude, is also a tangent to the other.*



From S, the ext. centre of simil. of ⊙s A, B, let SP be drawn, touching ⊙ A in P; and let BQ be ⊥ on SP (produced if necessary).

Then by simr. △s

$$AP : BQ = SA : SB = \text{radius of} \odot A : \text{radius of} \odot B.$$

$$\therefore \quad BQ = \text{radius of} \odot B.$$

$$\therefore \quad SPQ \text{ is tang. to} \odot B, \text{ at } Q.$$

The proof is exactly simr. for σ, the int. centre of simil.

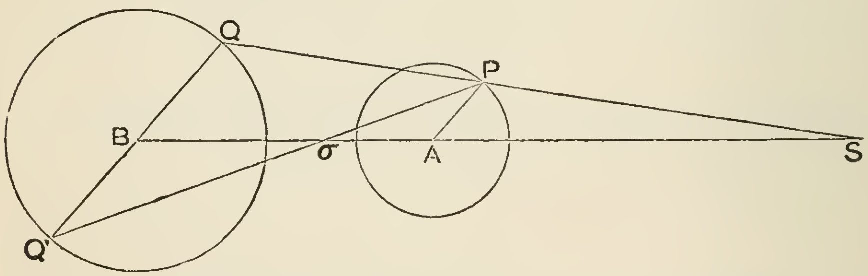
Def. The common tangents to two circles from their external centre of similitude are called their *direct common tangents*; and those from their internal centre of similitude are called their *transverse common tangents.*

Note—When one of the circles is entirely within the other, the circles have no common tangents, and therefore both centres of similitude lie within both circles.

Similarly when the circles intersect, the external centre of similitude lies outside both circles, and the internal centre of similitude within both.

And when the circles touch externally, the internal centre of similitude coincides with the point of contact.

THEOREM (2)—*The join of the extremities of two parallel radii of two circles goes through their external centre of similitude, when the radii are on the same side of the line of centres; and through their internal centre of similitude, when the radii are on opposite sides of the line of centres.*



Let AP, BQ be ‖ radii of ⊙s A, B; and on same side of AB.

Let QP, BA meet in S.

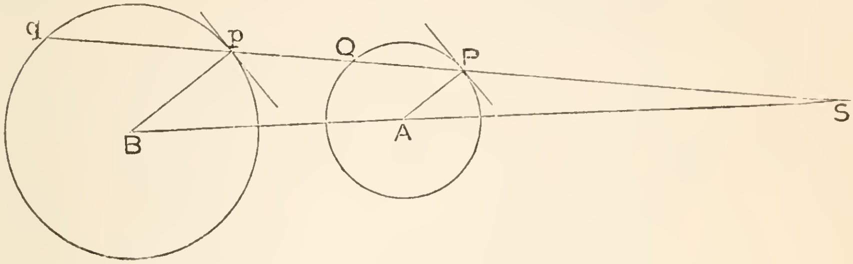
Then, by simr. △s, SA : SB = AP : BQ;

∴ S is *ext.* centre of simil.

Simrly. if AP, BQ′ are on opposite sides of AB; PQ′ will go through σ, the *int.* centre of similitude.

Def. If through either centre of similitude of two circles, a line is drawn cutting the circles; then, of the four points of section, the one which is nearer the centre of similitude on one circle is said to *correspond* to the one which is nearer on the other circle; and this pair of nearer points are called *corresponding points.* So also the pair of remoter points are called *corresponding points.* But a nearer point of one circle, and a remoter point of the other circle, are called *non-corresponding points.*

THEOREM (3)—*If through a centre of similitude of two circles, a line is drawn cutting the circles, the radii to a pair of corresponding points are parallel.*



Let S be a centre of simil. of ⊙s A, B; and let SPQpq cut ⊙ A in P, Q, and ⊙ B in p, q; so that P, p and Q, q are the pairs of corresponding points.

Then, since $SA : SB = AP : Bp$;

$\therefore$ AP and Bp are ∥.

Simrly. AQ and Bq are ∥.

Cor. The tangents at P, p, being ⊥ to AP, Bp are ∥; and so also are the tangs. at Q, q.

THEOREM (4)—*If through a centre of similitude of two circles, a line is drawn cutting the circles; then the rectangle under the distances of one pair of non-corresponding points from that centre of similitude, is equal to the rectangle under the distances of the other pair of non-corresponding points from that centre; and each of these rectangles is constant.*

Taking fig. of Theorem (3), we have

$$SP : Sp = AP : Bp = SQ : Sq;$$

$$\therefore \quad SP \,.\, Sq = SQ \,.\, Sp.$$

Also $SP \,.\, Sq : Sp \,.\, Sq = SP : Sp = SA : SB$, a const. ratio.

And $Sp \,.\, Sq$ = sq. on tang. from S to ⊙ B, which is const.

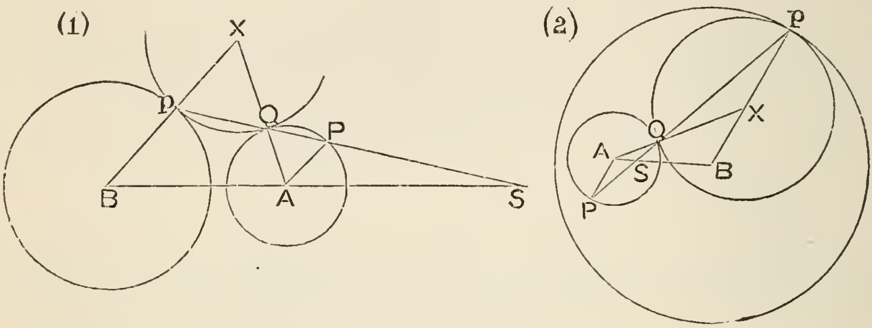
$\therefore$ $SP \,.\, Sq$ is const.

and $\therefore$ so also is $SQ \,.\, Sp$.

Simrly. for the internal centre of similitude.

Def. Each of these constant rectangles is called a *rectangle of anti-similitude.*

THEOREM (5)—*If a variable circle touches two fixed circles, the join of the points of contact goes through the external centre of similitude of the fixed circles, when the contacts are of the same kind; and through the internal centre, when of different kinds.*



Let ⊙ X be variable, and touch fixed ⊙s A, B, in Q, p respectively.

Let pQ cut BA in S, and ⊙ A again in P.

Then Bp, AQ will meet in X.

In fig. (1) contacts are of same kind.

In fig. (2) contacts are of opposite kinds.

In both figs. $X\hat{p}Q = X\hat{Q}p = A\hat{Q}P = A\hat{P}Q$.

$\therefore$ AP, Bp are ‖,

$\therefore$ AP : Bp = SA : SB;

$\therefore$ S is a centre of simil.

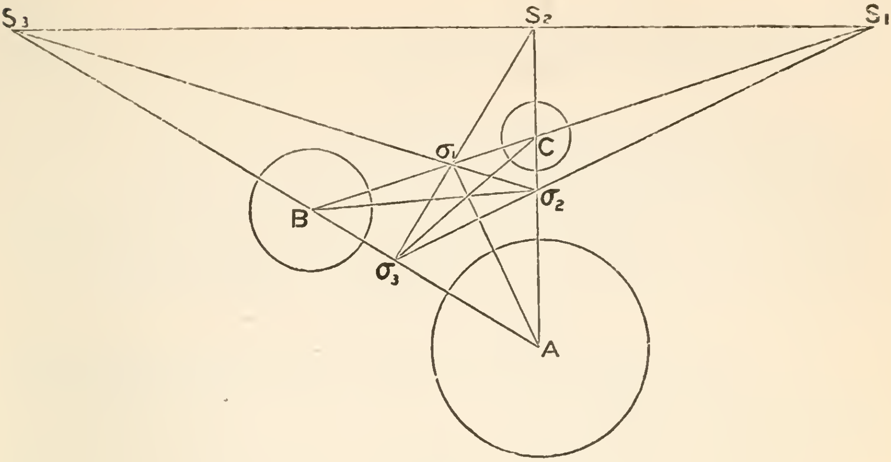
Cor. The tangent to ⊙ X from S is constant. For the sq. on it = Sp . SQ, which is const. by Theorem (4).

THEOREM (6)—*The six centres of similitude, which are got by taking three circles in pairs, are so situated that—*

(α) *the joins of the centre of each circle with the internal centre of similitude of the other two are concurrent:*

(β) *the external centre of similitude of any pair, and the two internal centres of similitude of the other two pairs, are collinear:*

(γ) *the three external centres of similitude are collinear.*



Let A, B, C be centres of three ⊙s;

$S_1\ \sigma_1$ the respective ext. and int. centres simil. of ⊙s B, C;

$S_2\ \sigma_2$ " " " C, A;

$S_3\ \sigma_3$ " " " A, B.

Let a, b, c be respective radii of ⊙s A, B, C.

Then $A\sigma_2 : \sigma_2 C = a : c$,

$C\sigma_1 : \sigma_1 B = c : b$,

$B\sigma_3 : \sigma_3 A = b : a$;

∴, compounding the ratios, we get

$(A\sigma_2 : \sigma_2 C)\ (C\sigma_1 : \sigma_1 B)\ (B\sigma_3 : \sigma_3 A) = 1$;

∴ $A\sigma_1$, $B\sigma_2$, $C\sigma_3$ are concurrent.

Again, $AS_2 : S_2 C = a : c$,

$CS_1 : S_1 B = c : b$,

$BS_3 : S_3 A = b : a$;

∴ $(AS_2 : S_2 C)\ (CS_1 : S_1 B)\ (BS_3 : S_3 A) = 1$;

∴ S_1, S_2, S_3, are collinear.

Simrly. it can be shown that

$S_1\ \sigma_2\ \sigma_3$ are collinear,

that $S_2\ \sigma_1\ \sigma_3$ "

and that $S_3\ \sigma_1\ \sigma_2$ "

Def. The line $S_1\ S_2\ S_3$ is called the *external axis of similitude* of the three circles; and the lines $S_1\ \sigma_2\ \sigma_3$, $S_2\ \sigma_1\ \sigma_3$, $S_3\ \sigma_1\ \sigma_2$ are called the three *internal axes of similitude.*

EXERCISES ON CENTRES OF SIMILITUDE.

1. The centroid and orthocentre of a triangle are respectively the internal and external centres of similitude of its circum-circle and N. P. circle.

2. D, E, F are the points of contact of the in-circle, respectively opposite the corners A, B, C of a triangle; if X is taken in CB, so that CX, BD are equal; and if AX cuts the in-circle in P, Q (of which P is nearer to A), then

$$AP \cdot BC = AE \cdot PX.$$

3. X, Y are the respective points of contact with BC, of the in-circle, and an ex-circle, of a triangle ABC; if YP is perpendicular to AX, then P is on the circumference of the ex-circle.

4. Through the external centre of similitude of two circles A, B, a variable line is drawn, meeting the circles in P, Q, p, q; if a circle is drawn touching A, B at non-corresponding points P, q; and another circle touching them at p, Q; then the difference of the radii of these last two circles is equal to the sum of the radii of A, B.

5. Given two non-intersecting circles; show that of all lines, parallel to a given direction, which meet the circles, the one through the internal centre of similitude has one of its segments, intercepted between the two circumferences, *maximum*, and one *minimum*.

NOTE—*Use Theorem* (3) *Cor.*

Def. The circle on $S\sigma$ [fig. of Theorem (1)] as diameter, is called the *circle of similitude* of circles A and B.

6. Show that the circle of similitude of two circles is the Locus of points at which the circles subtend equal angles.

NOTE—*By reference to* vi. *Addenda* (15) *it will be seen that the circle of similitude is such a Locus as is there investigated; and that if* X *is any point on its circumference,* XA : XB = *radius of* ⊙ A : *radius of* ⊙ B.

7. If PX, PY are tangents to two circles from any point P on their circle of similitude; and if XY meet the circles again in x, y; then will X x and Y y be equal. (*Chasles, Géométrie Supérieure,* p. 525.)

8. In Theorem (5) all the variable circles are cut orthogonally by a fixed circle.

9. Two fixed circles are each touched by two variable circles; if the variable circles also touch each other, find the Locus of their point of contact.

NOTE—*Use Theorem* (5).

10. If Tt is a common tangent, and $PQpq$ a common line of section of two circles, drawn through the same centre of similitude, and so that P, p and Q, q are corresponding points; then—

$$Tt^2 = Pp \cdot Qq.$$

11. If from S, a centre of similitude of two circles, two lines $SPQpq$, $SXYxy$, are drawn to cut the circles; so that P, Q, X, Y are on one circle; and p, q, x, y are the corresponding points on the other; and if each pair of points on the same circle are joined, then—

1°, the join of any pair, as PX, is parallel to the join px of the corresponding pair:

2°, the quadrilaterals $PYxq$, $QXyp$, $QYxp$, $PXyq$ are cyclic:

3°, the join of any pair of points, as PX, meets the join qy of the non-corresponding pair at a point such that the tangents from it to the circles are equal. [Cf. Section iv. (1) for the *Locus* of these points.]

12. If X is the centre of the circle of similitude of two circles A, B, whose respective radii are a, b, show that

$$AX : BX = a^2 : b^2.$$

Hence deduce that, if C is a third circle; and Y, Z the centres of the circles of similitude of B, C and of C, A; then X, Y, Z are collinear.

Def. The two circles round the two centres of similitude of a pair of circles as centres, the squares on whose radii are equal to the corresponding rectangles of anti-similitude, are called the *circles of anti-similitude.*

13. Every circle orthogonal to two circles is orthogonal at once to their circle of similitude, and their two circles of anti-similitude.

14. If A, B, C are any three circles; X a circle which touches them all internally, and Y a circle which touches them all externally; prove that—

1°, the radical axis (see Section iv.) of X, Y, is the axis of external similitude of A, B, C; and

2°, the internal centre of similitude of X, Y, is the radical centre of A, B, C.

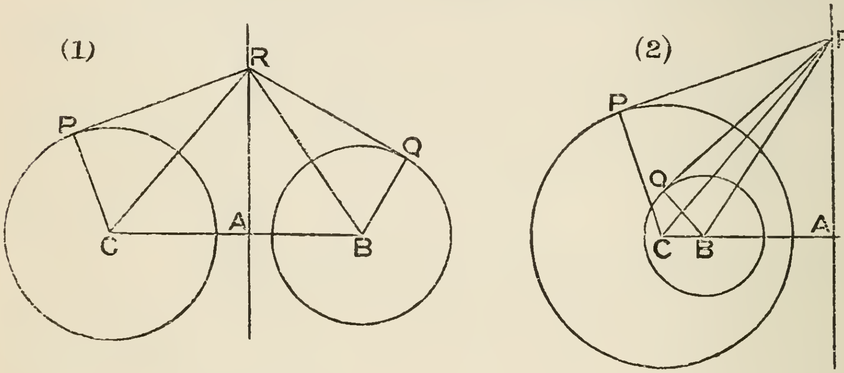
How should the Theorems be modified, when the contacts are not all of the same kind?

15. If a variable circle cuts two fixed circles, at equal angles, the join of a pair of non-corresponding points of intersection goes through the external centre of similitude of the fixed circles.

SECTION iv—CO-AXAL CIRCLES.

Def. That line which is perpendicular to the line of centres of two circles, and divides the distance between their centres (internally or externally) into segments, the difference of the squares on which is equal to the difference of the squares on the radii, is called the *radical axis* of the circles.

THEOREM (1)—*The radical axis of two circles is the Locus of points from which tangents to the circles are equal.*



Take any two ⊙s C, B—the radius of ⊙ C being the greater—and let A be the pt. in CB, fig. (1), or CB produced, fig. (2), for which

$$CA^2 - BA^2 = (\text{radius of} \odot C)^2 - (\text{radius of} \odot B)^2.$$

Let R be any pt. in the $\perp$ to CB at A.

Draw RP, RQ tangs. respectively to ⊙s C, B.

$$\text{Since} \quad CA^2 - BA^2 = CP^2 - BQ^2;$$

$$\therefore \quad CR^2 - BR^2 = CP^2 - BQ^2,$$

$$\text{or} \quad CR^2 - CP^2 = BR^2 - BQ^2;$$

$$\therefore \quad RP^2 = RQ^2.$$

i. e. tangs. from R to the ⊙s are equal;

and RA is the Locus of such points.

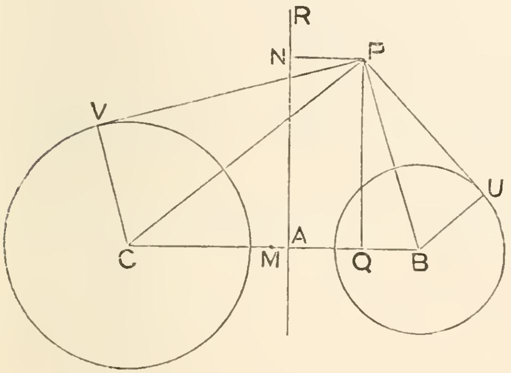
Cor. (1). When two ⊙s intersect, their radical axis is also their common chord.

Cor. (2). There is an unlimited number of $\odot^s$ having the same radical axis as any two $\odot^s$: for—

1°, when the $\odot^s$ intersect, *all* $\odot^s$ through the pts. of section have the same radical axis, by Cor. (1):

2°, otherwise; draw RS in *any* direction, on either side of RA, so that RS = RP: then if SX, $\perp$ to SR, meets CB in X, the $\odot$ with X as centre and XS as radius has the same radical axis as $\odot^s$ C, B.

THEOREM (2)—*The difference of the squares on tangents from any point to two circles, is equal to twice the rectangle under the join of their centres, and the distance of the point from their radical axis.*



Let RA be radical axis of $\odot^s$ C, B—A being in CB.

Let PV, PU be respective tangs. to them from any pt. P.

Draw PN, PQ $\perp^s$ respecty. to RA, CB.

Let M be the mid pt. of CB.

Then, since $PV^2 + CV^2 = PC^2$, and $PU^2 + BU^2 = PB^2$;

$$\therefore\ PV^2 - PU^2 + CV^2 - BU^2 = PC^2 - PB^2.$$

But $PC^2 - PB^2 = CQ^2 - BQ^2 = 2\,BC \,.\, QM$;

and $CV^2 - BU^2 = CA^2 - BA^2 = 2\,BC \,.\, AM$.

$$\therefore\ PV^2 - PU^2 = 2\,BC \,.\, QA = 2\,BC \,.\, PN.$$

Cor. (1). If P is on RA, PN = 0, and PV = PU, as in *Theorem* (1).

Cor. (2). If PU = 0, $PV^2 = 2\,BC \,.\, PN$.

Cor. (3). If PU = 0, and CV = 0, $PC^2 = 2\,BC \,.\, PN$. (Cf. p.191, *Ex.* 127.)

Cor. (4). If PXX′, PYY′ are secants, $PX \,.\, PX' - PY \,.\, PY' = 2\,BC \,.\, PN$.

THEOREM (3)—*The three radical axes of three circles (whose centres are not collinear) taken two and two, are concurrent.*

Let A, B, C be pts. not collinear; and let ⊙s A, B, C have respective radii r_1, r_2, r_3; also let X, Y, Z be respective pts. in which the three radical axes of the ⊙s cut BC, CA, AB.

$$\text{Then}\quad BX^2 - CX^2 = r_2^2 - r_3^2,$$
$$CY^2 - AY^2 = r_3^2 - r_1^2,$$
$$AZ^2 - BZ^2 = r_1^2 - r_2^2;$$
$$\therefore\quad (BX^2 - CX^2) + (CY^2 - AY^2) + (AZ^2 - BZ^2) = 0,$$

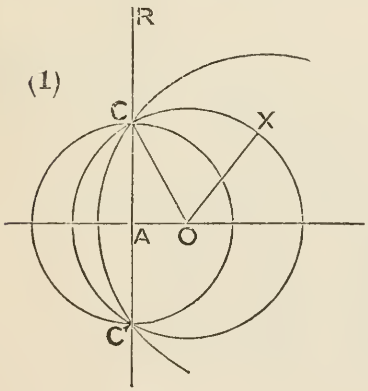
which is the condn. for the concurrency of the three ⊥s at X, Y, Z:

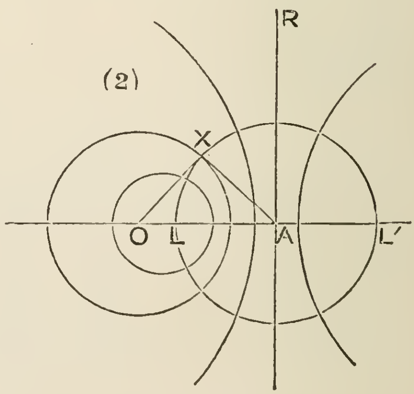
i.e. the three radical axes are concurrent.

Def. The point of concurrency of the three radical axes of three circles is called their **radical centre**.

Def. If a number of circles have their centres collinear, the line of centres is called their **central axis**.

Def. Any number of circles which have the same central axis, and the same radical axis, are said to be **co-axal**.



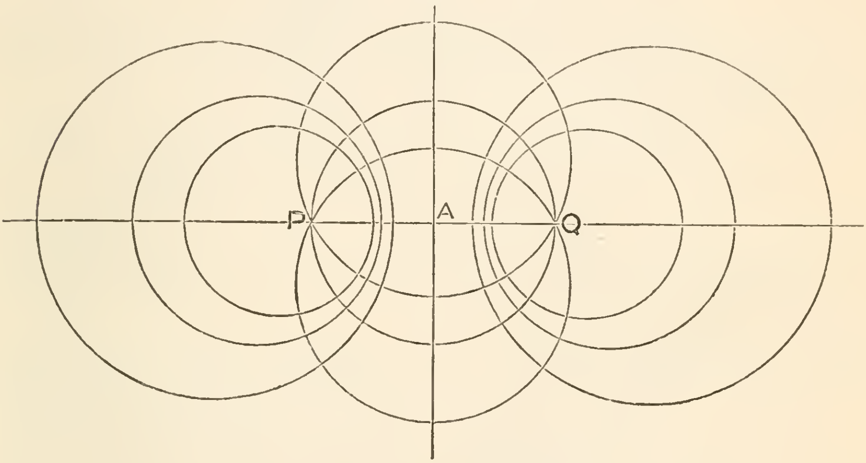


If O and OX are the variable centre and radius of any ⊙ of a co-axal system; and A the intersection of their central and radical axes; then, by *def.* of co-axal ⊙s, $OA^2 \sim OX^2$ is constant for all the system, $= \delta^2$ suppose.

1°, if any two of the ⊙s have common pts. C, C′, fig. (1), then all the ⊙s of the system go thro. C, C′; and the system is said to be of the *common point species.* In that case $CA = \delta = C'A$; and the ⊙ radius δ is *minimum.*

2°, if the ⊙s do not intersect, fig. (2), OX can diminish, until O comes to L, where AL = δ; and if we consider a pt. as the limiting value of a ⊙, when its radius decreases indefinitely, then L may be considered as the *minimum* ⊙; and if L′ is the image of L with respect to the radical axis, there are corresponding ⊙s, on the other side of the radical axis, with L′ as *minimum.* The two series of ⊙s together form a co-axal system which is said to be of the *limiting point species*—L and L′ being defined as *the limiting points.*

In neither species is there a *maximum circle* of the system; but the radical axis may be considered as the limit towards coincidence with which the ⊙s tend, as they increase indefinitely in size.



By consideration of the above figure, it will be seen that, if we take any two lines cutting at right angles in A; and P, Q are any points in one of them, equidistant from A; then P, Q will be the limiting points of one co-axal system, and the common points of another; and the lines are the corresponding central and radical, or radical and central, axes of the systems.

Hence to every system of the one species there is a corresponding system of the other species, such that the radical axis, central axis, and common (or limiting) points of the one, are respectively the central axis, radical axis, and limiting (or common) points of the other; and all ⊙s of one system are orthogonal to all ⊙s of the other system.

Suppose δ to diminish indefinitely: then the limiting (or common) points approach indefinitely near A; and will coincide with A, when δ vanishes.

Hence, if a series of ⊙s are in contact at one point, the series may be considered as a co-axal system of which the point of contact is a coincident position of the two limiting, or the two common, points of the system.

If ∴ it can be shown that the limiting point of two ⊙s is *on* one of them, it follows that the ⊙s *touch* at that point.

THEOREM (4)—*If three circles are co-axal, the squares on tangents to two of them from a point on the third, are in the ratio of the distances of the centre of the third from the other two centres.*

Let ⊙s A, B, C be co-axal; P a pt. on ⊙ C; PT, PS tangs. to ⊙s A, B; PN ⊥ on their radical axis. Then, by *Theorem* (2) *Cor.* (2),

$$PT^2 = 2\,CA \,.\, PN, \text{ and } PS^2 = 2\,CB \,.\, PN;$$

$$\therefore\ PT^2 : PS^2 = CA : CB.$$

Cor. If ⊙ B is supposed to shrink up into a limiting pt. L of the system,

$$\text{then}\quad PT^2 : PL^2 = CA : CL$$

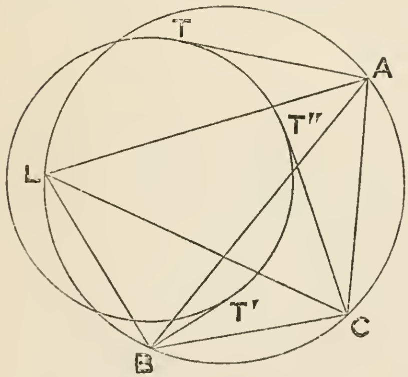
∴, if L is a limiting pt. of two ⊙s, and PT a tang. from any pt. P on the one ⊙ to the other, then PT : PL is a constant ratio.

And, conversely, if P, P′, P″, L, are four pts.; PT, P′T′, P″T″ tangs. to a ⊙;

$$\text{and if}\quad PT : PL = P'T' : P'L = P''T'' : P''L,$$

then L is a limiting pt. of this ⊙ and the ⊙ round PP′P″.

THEOREM (5)—*(Feuerbach's) The nine-point-circle of a triangle touches its in-circle, and each of its three ex-circles.*



LEMMA—If AT, BT′, CT″ are tangs. to a ⊙ from the pts. A, B, C; and if, of the three rects. BC . AT, CA . BT′, AB . CT″, the sum of any two = the third, then the ⊙ will touch the circum-⊙ of △ ABC.

For, if AB . CT″

= BC . AT + CA . BT′,

let L be the pt. on the side of AB remote from C, where the *Apollonius' Locus* (*cf.* p. 294) given by

$$AL : BL = AT : BT'$$

cuts the circum. of △ ABC.

$$\therefore\ BC \,.\, AT : CA \,.\, BT' = BC \,.\, AL : CA \,.\, BL$$

∴, *componendo et alternando,*

$$BC \,.\, AT + CA \,.\, BT' : BC \,.\, AL + CA \,.\, BL = BT' : BL = AT : AL$$

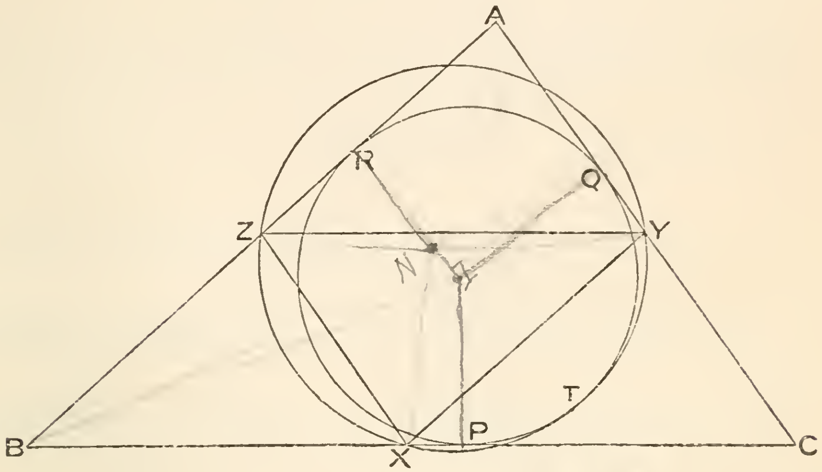
But, by the hypothesis and *Ptolemy's Theorem,*

$$BC \,.\, AT + CA \,.\, BT' : BC \,.\, AL + CA \,.\, BL = AB \,.\, CT'' : AB \,.\, CL$$
$$= CT'' : CL$$

$$\therefore\ AT : AL = BT' : BL = CT'' : CL$$

$\therefore$ L is a limiting pt. of the $\odot^s$.

$\therefore$, as L is on one of the $\odot^s$, the $\odot^s$ touch at L.



Now let a, b, c be the sides; X, Y, Z the mid pts. of those sides; P, Q, R their pts. of contact with the in-$\odot$; in each case respectively opposite the corners A, B, C of a $\triangle$.

Then the circum-$\odot$ of $\triangle$ XYZ is the N. P. $\odot$ of $\triangle$ ABC.

Also (*Cor.* p. 211) $2\,XP = b \sim c$, $2\,YQ = c \sim a$, $2\,ZR = a \sim b$

And $2\,YZ = a$, $2\,ZX = b$, $2\,XY = c$

But, whatever is the order of magnitude of a, b, c, the expressions

$$ab \sim ac, \quad bc \sim ab, \quad ac \sim bc,$$

are so related that the sum of any two = the third.

$\therefore$ the same is true of $YZ \cdot XP$, $ZX \cdot YQ$, $XY \cdot ZR$

$\therefore$, by the **Lemma**, the N. P. $\odot$ touches the in-$\odot$ at a pt. T.*

In a precisely similar way the N.P. $\odot$ can be shown to touch each of the ex-$\odot^s$.

* The above (due to Professor Purser, of Queen's College, Belfast, by whom it was communicated to the Editor) is by far the most easily recollected of the many proofs of this famous Theorem.

EXERCISES ON CO-AXAL CIRCLES.

1. The circle of similitude of two circles is co-axal with them.

2. The three circles of similitude of three circles, taken two and two, are co-axal.

3. Given two intersecting circles A, B; show that there is another circle C, co-axal with them, such that, if tangents are drawn to the three from any point, then—sq. on tang. to A + sq. on tang. to B = 2 sq. on tang. to C.

4. A variable circle cuts two fixed circles orthogonally; find the Locus of its centre.

5. If a variable circle touches two fixed circles, its radius bears a constant ratio to the distance of its centre from their radical axis.

6. A variable circle goes through a fixed point A, and cuts a fixed circle orthogonally in P, Q: show that the rectangle under AP, AQ varies as the chord PQ.

7. Given a fixed circle, centre C, and a fixed line AB; if a system of circles have AB as their central axis, and cut given circle orthogonally, they are co-axal, their radical axis being the perpendicular from C on AB.

8. In a co-axal system, of the limiting point species, if through a limiting point L, a line is drawn to cut a circle of the system in X, Y; and if XM, YN are perpendiculars on the radical axis, then XM . YN = LA^2, where A is the intersection of the radical and central axes.

9. Perpendiculars drawn through the mid points of the sides of a triangle to the bisectors of its angles, are the radical axes of the in-circle and the several ex-circles.

10. X, Y, P, Q, R are any five circles; if ABC is the triangle formed by the radical axes of X, P, of X, Q, and of X, R; and $\alpha\beta\gamma$ is the triangle formed by the radical axes of Y, P, of Y, Q, and of Y, R; then ABC, $\alpha\beta\gamma$ are in perspective.

11. If AOD, BOE, COF are the altitudes of a triangle ABC, and G its centroid; then the circum-circles of ADG, BEG, CFG, have a second point in common; namely, the intersection of OG with the radical axis of the N. P. circle and the circum-circle of ABC.

12. The sixteen circles of contact, of the four triangles formed by four intersecting lines, have their centres, in fours, on four co-axal circles.

13. Each of the four triangles, formed by joining the four points of contact of the N. P. circle with the in- and ex- circles, is in perspective with the original triangle.

SECTION v—"THE TANGENCIES."

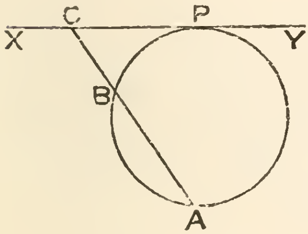
Given in position any three of the following nine—three points, three lines, three circles of fixed magnitude—it is required to describe a circle which shall pass through such points as are given, and touch such lines and circles as are given. These Problems are known as *The Tangencies.*

There are *ten* cases.

Case 1. *Given three points*—Euc. iv. 5.

Case 2. *Given three lines*—Euc. iv. 4.

Case 3. *Given two points and a line.*



Let A, B be given pts. and XY given line.

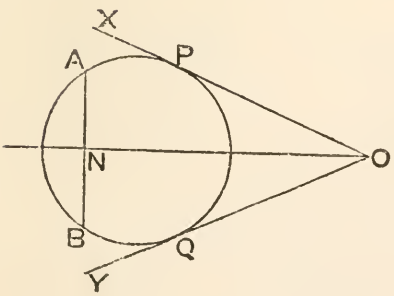
Join AB, and let it meet XY in C.

In XY take CP a mean propl. to CA, CB; so that $CP^2 = CA \cdot CB$.

Then the ⊙ thro. A, B, P touches XY at P.

There are two positions of P, and ∴ two solutions.

Case 4. *Given two lines and a point.*



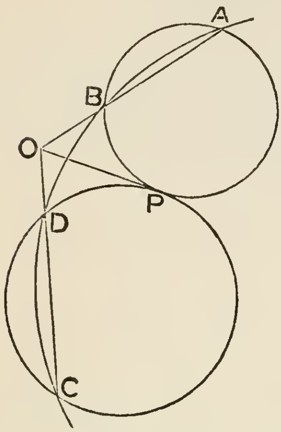
Let OX, OY be the given lines; A the given pt.

Draw ON the bisector of $\widehat{XOY}$; and let B be the image of A with respect to ON.

By *Case* 3, draw a ⊙ thro. A, B to touch OX in P: then this ⊙ will clearly touch OY in Q, the image of P with respect to ON.

There are two positions of P, and ∴ two solutions.

Case 5. *Given two points and a circle.*



Let A, B be the given pts.

Draw any ⊙ thro. A, B to cut given ⊙ in C, D.

Let AB, CD meet in O; and draw OP to touch given ⊙ in P.

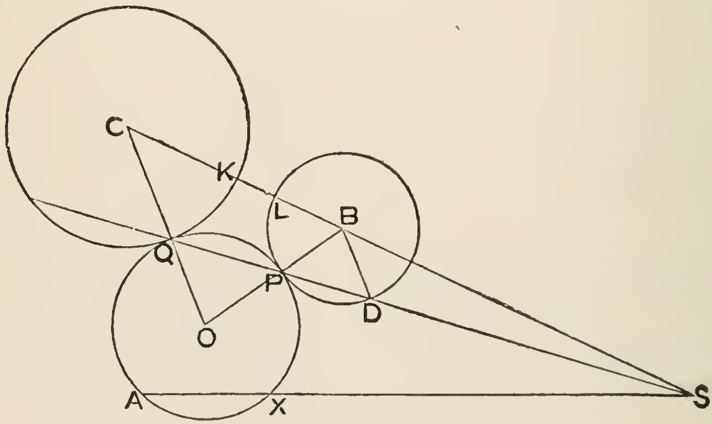
Then, since $OP^2 = OC \,.\, OD = OA \,.\, OB$,

∴ ⊙ thro. A, B, P touches OP in P;

and ∴ also touches given ⊙ in P.

There are two positions of P, and ∴ two solutions.

Case 6. *Given two circles and a point.*



Let A be given pt.; and B, C centres of given ⊙s.

Take S the ext. centre of similitude of B, C.

Let BC cut ⊙s in L, K.

Join SA; and in it take SX the fourth propl. to SA, SK, SL, so that

$$SA : SK = SL : SX,$$

$$\text{and} \therefore\ SA \,.\, SX = SK \,.\, SL.$$

Thro. A and X describe (*Case* 5) ⊙ to touch ⊙ B in P; let O be its centre.

Join SP, cutting ⊙ B in D again, and meeting ⊙ C in Q.

Then $SA \,.\, SX = SK \,.\, SL = SP \,.\, SQ$, ∵ S is cent. of simil.

∴ Q is on ⊙ thro. A, X, P.

Also BD is ∥ to CQ, by a property of centre of similitude.

$$\therefore\ \widehat{SDB} = \widehat{CQS}.$$

$$\text{But}\ \widehat{OQP} = \widehat{OPQ} = \widehat{BPD} = \widehat{BDP} = \text{suppt.}\ \widehat{SDB};$$

$$\therefore\ \widehat{OQP} = \text{suppt.}\ \widehat{CQS};$$

∴ CQO is a st. line;

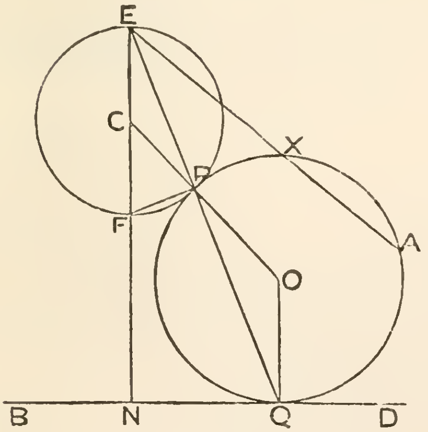
∴ ⊙ O touches ⊙s B, C, and goes thro. A.

Since there are two positions of P, there are two solutions corresponding to S.

Also taking the internal centre of similitude there will be two more solutions.

i.e. there are four solutions in all.

Case 7. *Given a point, a line, and a circle.*



Let A be given pt.; C centre of given ⊙; and BD given line.

Draw CN ⊥ to BD at N, and let it meet ⊙ in F.

Produce NC to meet ⊙ again in E.

Join EA; and in it take EX the fourth propl. to EA, EN, EF; so that

$$EA : EN = EF : EX,$$

$$\text{and} \therefore\ EN \,.\, EF = EA \,.\, EX.$$

Thro. A and X describe (*Case* 3) a ⊙, centre O, to touch BD in Q.

Join EQ, cutting given ⊙ in P; and join CP, OP, OQ, FP.

Then since $\widehat{FPE}$ and $\widehat{FNQ}$ are each right,

$\therefore$ quad. FNQP is cyclic.

$\therefore$ EP . EQ = EF . EN = EA . EX;

$\therefore$ P is on circumf. of ⊙ O.

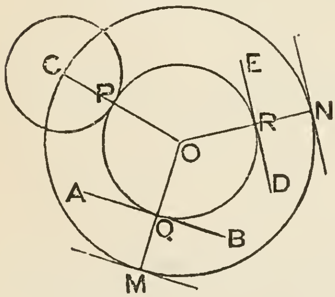
$\therefore \widehat{OPQ} = \widehat{OQP} = \widehat{CEP} = \widehat{CPE}$;

$\therefore$ CPO is a st. line.

$\therefore$ ⊙s touch at P.

Two ⊙s can be described thro. A and X, touching BD, which give two solutions; and similarly two more solutions can be got by joining FA, instead of EA.

Case 8. *Given two lines, and a circle.*



Let AB, DE be given lines; C centre of given ⊙.

On sides of AB, DE, remote from C, draw ||s to them, at distance from them which = rad. of C.

Describe (*Case* 4) a ⊙ thro. C to touch the || to AB in M, and the || to DE in N.

Let O be its centre; and let OM cut AB in Q, and ON cut DE in R.

Join CO cutting given ⊙ in P; and with centre O and radius OP describe a ⊙.

Then OC = OM, and QM = CP.

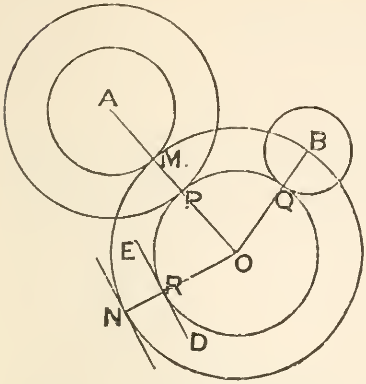
$\therefore$ OQ = OP, which simrly. = OR.

$\therefore$ Q and R are pts. on ⊙ O.

Also ⊙ touches AB, DE at these pts, $\because$ $\wedge$s there are right.

The two solutions of *Case* 4 give two solutions: two more will be got by drawing ||s to AB, ED on *same* side as C.

Case 9. *Given two circles and a line.*



Let A, B be centres of given ⊙s; DE given line.

On side of ED remote from B (the centre of the lesser ⊙) draw a ‖ to DE at distance from it which = rad. of B.

With centre A, and radius which = diff. of radii of A and B, describe a ⊙.

Draw (*Case* 7) a ⊙ thro. B to touch the constructed line at N, and constructed ⊙ at M.

Let O be centre of this ⊙; let ON cut ED in R; let OA cut ⊙ A in P; and let OB cut ⊙ B in Q.

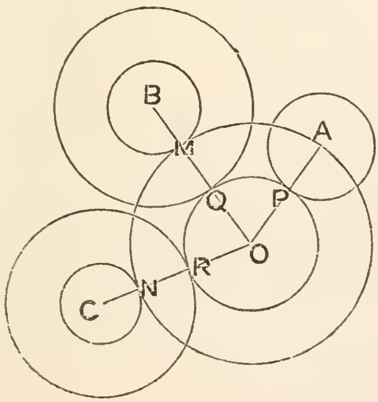
Then since OB = ON, and QB = RN,

∴ OR = OQ, which simrly. = OP.

∴ ⊙ centre O and rad. OP goes thro. Q and R, and touches given ⊙s and line at P, Q, R.

The four solutions of *Case* 7 will give four solutions here: four more will come from drawing ‖ to ED on *same* side as B.

Case 10. *Given three circles.*



Let A, B, C be centres of given ⊙s, of which A has a radius which ≯ that of either of the others.

With B as centre, and radius which = rad. B − rad. A, describe a ⊙.

With C as centre, and radius which = rad. C − rad. A, describe a ⊙.

Thro. A draw a $\odot$ to touch these last two constructed $\odot$s (*Case* 6) in M, N respectively.

Let BM, CN meet in O; and join OA.

Let OA cut $\odot$ A in P; OB cut $\odot$ B in Q; and OC cut $\odot$ C in R.

Then OM = OA, and MQ = PA.

$\therefore$ OQ = OP, which simrly. = OR.

$\therefore$ $\odot$, centre O radius OP, goes thro. Q and R; and touches given $\odot$s in P, Q, R, $\because$ OA, OB, OC are lines of centres.

It will easily appear that there are eight solutions.

EXERCISES ON THE TANGENCIES.

1. For certain relations among the *data*, some of the preceding constructions fail: investigate the necessary modifications to be made, when in—

Case (3), AB is parallel to XY; or A is in XY:

Case (4), given lines are parallel; or A is in one of them:

Case (5), A and B are equidistant from centre of given circle; or A is on given circle:

Case (6), given circles are equal.

2. In the figure of *Euc.* i. 1, describe a circle to touch the given line, and the two circles of construction.

3. In *Case* (7) there are generally two circles which can be drawn to have external contact with the given circle: if A is supposed variable, find its Locus under the condition that these two circles touch each other.

4. Given three circles, describe another to touch two of them, and—

1°, bisect the circumference of the third;

2°, cut the third orthogonally.

5. If we consider a point as an infinitely small circle, and a line as an infinitely large circle; show that *Cases* (3) to (10) may all be solved by the following construction (*Gergonne's*)—Let A, B, C be three given circles; O their orthogonal circle: let the chords of intersection of O with A, B, C meet an axis of similitude of A, B, C in P, Q, R; and from P, Q, R draw pairs of tangents to A, B, C respectively: then the two circles through the six points of contact of these tangents will touch A, B, C: also, since there are four axes of similitude, there will be eight circles of contact.

SECTION vi—INVERSION.

Def. Every two points P and Q, on a diameter of a circle (centre C) such that the rectangle under CP, CQ is equal to the square on the radius, are called **inverse points** with respect to that circle. Also the circle is called the **circle of inversion**; and its centre is called the **centre of inversion.**

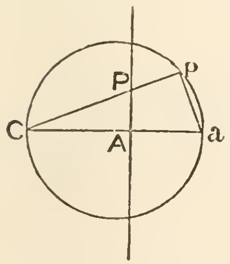
Note—Any fixed circle may be taken as the circle of inversion.

Def. The **inverse of a Locus** is the Locus of the inverses of all points on it.

Thus if to every position of a point P on a Locus we take the corresponding inverse p, then the Locus of p is the inverse of the Locus of P.

Note—In what follows the radius of the circle of inversion is denoted by R.

THEOREM (1)—*The inverse of a line is a circle through the centre of inversion.*



Let AP be a line; C the centre of inversion; CA $\perp$ to AP; and P any pt. in AP.

Let a be inverse of A, and p inverse of P.

$$\text{Then} \quad CP \cdot Cp = R^2 = CA \cdot Ca,$$

$\therefore$ P, p, a, A are concyclic.

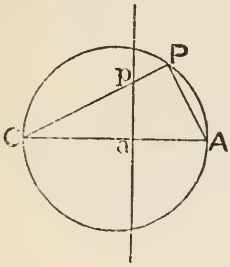
But $P\hat{A}a$ is right.

$\therefore$ $C\hat{p}a$ is right.

$\therefore$ Locus of p is $\odot$ on Ca as diameter.

Cor. A line is the radical axis of its inverse and the circle of inversion.

THEOREM (2)—*The inverse of a circle through the centre of inversion is a line.*



Let CA be diam. of ⊙ thro. C, the centre of inversion; P any pt. on this ⊙.

Let a be inverse of A, and p inverse of P.

$$\text{Then}\quad CP \,.\, Cp = R^2 = CA \,.\, Ca;$$

$\therefore$ P, p, a, A are concyclic.

But $C\hat{P}A$ is right.

$\therefore$ $C\hat{a}p$ is right.

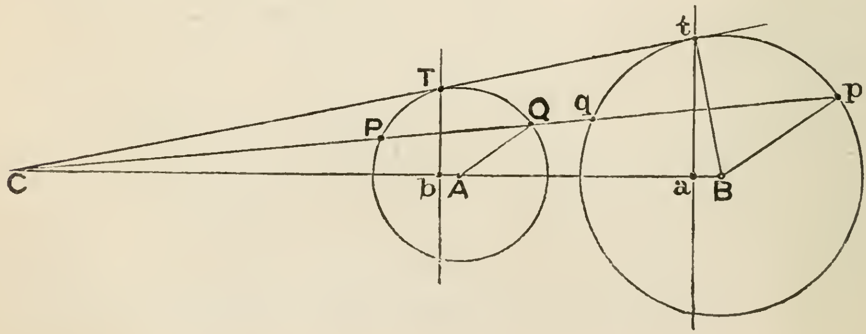
$\therefore$ Locus of p is the line $\perp$ to CA.

Note—We can now give the theory of the *Peaucellier* movement: for, referring to the figure on p. 5, by the symmetry of the instrument, P, O, Q will always be in one line; and if ANB cuts this line in N, the $\wedge^s$ at N are right, and N is the mid point of the diags. of the rhombus.

$$\therefore\quad QO \,.\, PO = (QN + NO)\,(QN - NO),$$
$$= QN^2 - NO^2,$$
$$= QA^2 - AO^2, \text{ which is const.}$$

$\therefore$ P, Q are inverse points; and, since Q moves on a ⊙ thro. the centre of inversion O, P will move on a st. line.

THEOREM (3)—*The inverse of a circle* not *through the centre of inversion is a circle.*



Let A be centre of given ⊙, P any pt. on it, and C the centre of inversion. Let CP cut the ⊙ again in Q; and let p, q be the respective inverses of P, Q.

$$\therefore \quad CP \,.\, Cp = R^2 = CQ \,.\, Cq$$

Join AQ; and draw pB ‖ to QA to meet CA in B.

Then CP . CQ = (tang. from C)^2, and ∴ is const.

∴ Cp : CQ (which = CP . Cp : CP . CQ) also is const.

∴ CB : CA and Bp : AQ are each const.

∴ B is a fixed pt., and Bp of fixed length.

∴ the Locus of p is a ⊙ centre B.

Cor. (1). The centre of inversion is a centre of similitude of a ⊙ and its inverse; and (the rad. of inversion)^2 = either rect. of anti-similitude.

Cor. (2). If CTt touches ⊙ A in T, and ⊙ B in t; then

$$\text{rad. of } \odot A : \text{rad. of } \odot B = CT^2 : R^2 = R^2 : Ct^2$$

Cor. (3). If Tb, ta are ⊥s on CAB, meeting it in b, a; then ta, Tb are respectively the inverses of the ⊙s on CA, CB as diams.

∴ a is the inverse of A, and b is the inverse of B.

Hence, since BC . Ba = Bt^2, we see that—*The inverse of the centre of a circle is the inverse of the centre of inversion with respect to the inverse circle.*

THEOREM (4)—*Each point of intersection of two Loci is the inverse of a point of their inverse Loci.*

For the inverse of a pt. of section of the Loci must be a pt. on *each* of the inverse Loci: i. e. must be *a* pt. where they intersect.

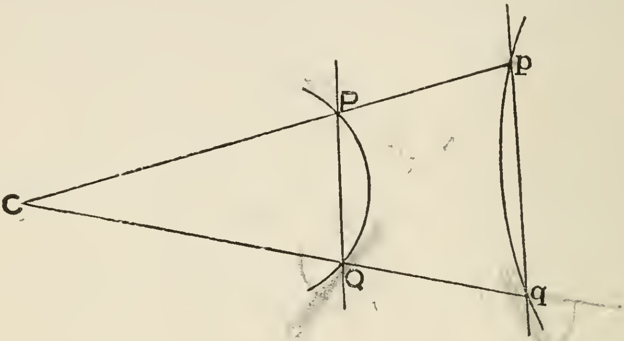
Def. If one of the points of section of a secant of a circle is made to move up to the other, then the limiting position of the secant (to which it constantly approaches, and which it ultimately assumes, when the points are brought indefinitely near together) is called a **tangent** to the circle.

Note—This definition of a tangent will be seen to amount to the same as Euclid's, if we consider that it may be put thus—a tangent is a secant through two coincident points, that is through one point.

Def. The angle between a line and a circle, is the angle made by the line with the tangent at the point where the line cuts the circle; and the angle between two circles, is the angle between their tangents at a point of section.

Def. A variable line from a fixed point to a fixed circle is called a **radius vector** of the circle, with respect to that point as *origin.*

THEOREM (5)—*Any radius vector of a circle, with respect to the centre of inversion as origin, cuts the circle and its inverse at supplementary angles—the angles being measured by rotation, in an anti-clockwise manner, from the radius vector as initial position.*



Let CPp, CQq be any radii vectores thro. C, the centre of inversion; so that p, q are respective inverses of P, Q.

$$\text{Then} \quad CP \cdot Cp = R^2 = CQ \cdot Cq,$$

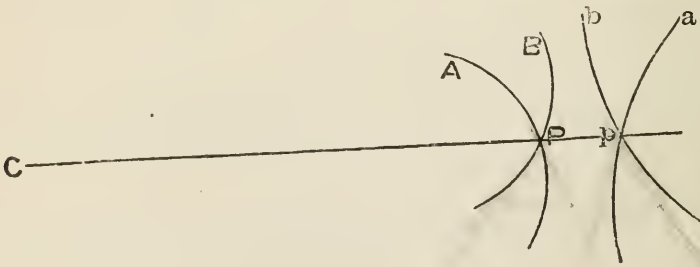
$$\therefore \quad P, Q, q, p \text{ are concyclic.}$$

$$\therefore \quad P\hat{Q}q = \text{suppt. } P\hat{p}q.$$

But when Q moves up to P, so does q up to p.

And, ultimately, when Q coincides with P, PQ becomes the tangent at P; and pq the tangent at p; and then PQq becomes the $\wedge$, and Ppq vertically opposite to the $\wedge$ (measured according to the convention above) at which CPp cuts the $\odot$ and its inverse respectively.

THEOREM (6)—*Two Loci cut at the same angle as their inverses cut.*



Let A, B be two Loci, which cut in P; a, b their inverses, which cut in p; so that p is inverse of P.

Then CPp is a radius vector, where C is the centre of inversion.

$\therefore$ $\wedge$ between CP, AP = suppt. $\wedge$ between Cp, ap.

And $\wedge$ between CP, BP = suppt. $\wedge$ between CP, bp.

But diff. of two $\wedge^s$ = diff. of their suppts.

$\therefore$ $\wedge$ between AP, BP = $\wedge$ between ap, bp.

Cor. (1). If two circles (or a line and a circle) touch, their inverses touch.

Cor. (2). If a Locus and its Envelope are inverted, their inverses touch.

Cor. (3). If two Loci cut orthogonally, so do their inverses.

THEOREM (7)—*If* a, b, c, &c., *are the respective inverses of any number of points* A, B, C, &c.; *then* (O *being the centre of inversion*)—

1°, *for every two points* A, B,

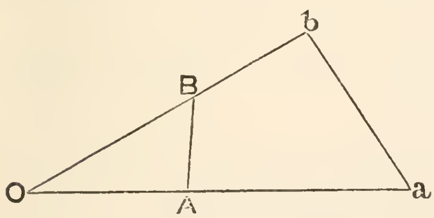
$$ab : AB = R^2 : OA \cdot OB.$$

2°, *for every three points* A, B, C,

$$bc : ca = OA \cdot BC : OB \cdot CA.$$

3°, *for every four points* A, B, C, D,

$$bc \cdot ad : ca \cdot bd = BC \cdot AD : CA \cdot BD.$$



For since
OA . Oa = R^2 = OB . Ob,
$\therefore$ Δ^s OAB, Oba are simr.

$\therefore$ 1°, ab : AB = Oa : OB,

= OA . Oa : OA . OB,

= R^2 : OA . OB.

2° and 3° follow at once from this.

Cor. Cross-ratios are unchanged by inversion.

The preceding Theorems are fundamental. From them may be easily deduced the following, which are important—

(8)—*Two points and their inverses are concyclic.*

(9)—*If* A, B *are points, and* a, b *their inverses; then*—O *being the centre of inversion*—

$$ab : AB = \perp \textit{ from } O \textit{ on } ab : \perp \textit{ from } O \textit{ on } AB.$$

(10)—*A circle through a pair of inverse points, cuts diameters of the circle of inversion in inverse points.*

(11)—*Any point on a chord of a circle has its inverse (with respect to that circle) concyclic with the centre and the extremities of the chord.*

(12)—*Every circle through a pair of inverse points, on same side of the centre of inversion, is orthogonal to the circle of inversion.*

(13)—*If a circle is inverted with respect to any orthogonal circle, it is its own inverse; and it is said to invert into itself.* So also—*A circle inverts into itself if the mid point of a chord, and the semi-chord are taken as centre, and radius of inversion.*

(14) *Every two circles invert each into the other, with respect to* either *of their two circles of anti-similitude.* (Cf. Exercise 12, p. 345.)

(15)—*Of two orthogonal circles, either cuts diameters of the other in points which are inverses with respect to that other.*

(16)—*If* P, Q *are inverse points,* C *the centre of inversion, and* X *any point on the circle of inversion; then*—

$$PX^2 : QX^2 = PC : QC;$$

and conversely.

NOTE—*Follows from similarity of* Δs PCX, XCQ.

(17)—*If* L *is the line with respect to which* P, Q *are images, each of the other, then*—

$$PX^2 = 2\,PC \,.\, XL, \quad \textit{and} \quad QX^2 = 2\,QC \,.\, XL;$$

and conversely—XL *being perpendicular to line* L.

(18)—*Two circles, inverses of each other, are co-axal with the circle of inversion.*

(19)—*Every two circles invert into equal circles from every point on either of the co-axal circles, whose centres are their centres of similitude.*

(20)—*In a co-axal system of circles, of the* **limiting-point species,** *the limiting points are inverses with respect to every circle of the system.*

(21)—*If a co-axal system, of the* **common-point species,** *is inverted with respect to either common point as centre of inversion, the inverse system consists of lines concurrent in the inverse of the other common point.*

(22)—*If a co-axal system, of the* **limiting-point species,** *is inverted with respect to either limiting point, the inverse system is a series of concentric circles, whose centre is the inverse of the other limiting point.*

As a striking example of the last Theorem—Let one circle lie within another: then if *any one complete ring of circles* can be placed between them (touching them and each other) an infinite number of such rings can be placed.

This follows at once by inverting the circles with respect to a limiting-point as centre of inversion.

EXERCISES ON INVERSION.

1. Show that—

(1) *Euc.* i. 13 inverts into *Euc.* iii. 22;

(2) ,, i. 29 ,, ,, iii. 32;

(3) ,, i. 5, 6 ,, ,, vi. 3;

(4) ,, iii. 27 ,, ,, vi. 3;

(5) *Euler's Theorem*, p. 104, inverts into *Ptolemy's Theorem*, p. 289.

(6) *Euc.* i. 20 inverts into *Euc.* vi. *Addenda* (9).

2. What does *Ptolemy's Theorem* invert into, when the centre of inversion is—

1°, a corner of the quadrilateral?

2°, a point on the circle round the quadrilateral, but *not* a corner?

3. Show that Locus (δ), p. 178, inverts into vi. *Addenda* (15).

4. Show that if Locus (ι), p. 178, is inverted with respect to one of the extremities of the fixed base, it gives the Theorem—If the base of a triangle, and ratio of its area to the area of the square on one of its sides, are given; then the Locus of its vertex is a circle, touching its base at one end.

5. If A, B, C are fixed collinear points; and P a variable point such that the angles APB, BCP are equal; then evidently the Locus of P is a circle, centre A: show that if this is inverted with respect to a point in CBA produced, the inverse Theorem is Locus vi. *Addenda* (17).

6. What is the inverse of a polygon?

7. Invert vi. *Addenda* (13).

8. Invert *Euc.* iii. 35, when C is the mid point of the arc AB.

9. A, O, B, P are concyclic points: if A, O, B are fixed, and P variable, what Theorem results from inverting the figure with respect to O?*

10. Invert the characteristic property of *Simson's Line*, p. 172.

11. If there are three fixed circles through two fixed points, and any variable circle is drawn, cutting two of them at fixed angles, it will cut the third at a fixed angle.

NOTE—*Invert with respect to one of the fixed points.*

12. If a quadrilateral ABCD is *not* cyclic, prove that the rectangles AB . CD, BC . AD, CA . BD, are proportional to the sides of a triangle, of which an angle is equal to the sum of two opposite angles of the quadrilateral.

NOTE—*Invert with respect to one corner of the quad.*

13. From a fixed point O, variable lines OX, OY are drawn to meet a fixed line in X, Y; and so that the angle XOY is constant: show that the circum-circle of triangle XOY always touches a fixed circle. (*Messenger of Mathematics*, Vol. III. p. 231.)

NOTE—*Invert with respect to* O.

14. Show that every two figures, inverses of each other, invert from any point into two figures, inverses of each other with respect to the inverse of the original circle of inversion.

What does the Theorem become when the point is on the circle of inversion?

15. Show that *Feuerbach's Theorem* [Section iv, *Theorem* (6)] may be proved by inversion thus—Let Q be point of contact of in-circle, centre I; and Q′ of ex-circle, centre E; M the mid point of BC and QQ′: take M and MQ as centre and radius of inversion. Then circles I, E invert into themselves, and circle N (nine-point) inverts into a line perpendicular to MN, cutting BC in R, where $MR . MP = MQ^2$: P being the foot of the altitude from A. (*J. P. Taylor*, *Quarterly Journal*, Vol. XIII. p. 197.)

NOTE—*Show that* R *is on* IE; *and that the line inverse of* ⊙ N *is the fourth common tangent of* ⊙s I, E.

* Exercises 9, 8, 7, 5, 4, 3, and (1), (2), (3) of 1, are due to Mr. R. A. H. MacFarland of Caius College, Cambridge.

SECTION vii—HARMONIC RANGES.

Def. If the segment AB of a line is divided internally in X and externally in Y, in the same ratio, so that

$$AX : BX = AY : BY,$$

then the four points A, X, B, Y are termed a **harmonic range**; and the pair of points X and Y are termed **harmonic conjugates** of each other with respect to A and B.

Note (1)—Since the above relation may be written

$$YB : XB = YA : XA,$$

it follows that A and B are harmonic conjugates of each other with respect to X and Y.

Note (2)—Since AX . BY = BX . AY, the cross-ratio (AXBY) is unity, and ∴ constant. Hence all Theorems deduced from the constancy of cross-ratios are true for harmonic ranges.

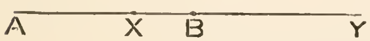
Def. Three magnitudes are said to be in **harmonic proportion** when

$$\text{1st} : \text{3rd} = \text{1st} \sim \text{2nd} : \text{2nd} \sim \text{3rd}.$$

And the 2nd is termed the **harmonic mean** between the 1st and 3rd.

The words *harmonic range* and *harmonic mean* will be respectively abbreviated into H. R. and H. M.

Theorem (1)—*If four points form an* H. R. *the distance of either extreme point from its own conjugate is an* H. M. *between its distances from the other two.*



For if A, X, B, Y is an H. R.,

then $YA : YB = XA : XB = YA - YX : YX - YB,$

and ∴ YX is an H. M. between YA, YB.

And again, $AY : AX = BY : BX = AY - AB : AB - AX,$

and ∴ AB is an H. M. between AX, AY.

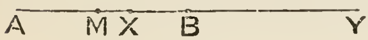
Cor. Since $AY (AB - AX) = AX (AY - AB)$;

$\therefore \quad AB (AX + AY) = 2\,AX \,.\, AY.$

Simrly. $XY (AY + BY) = 2\,AY \,.\, BY.$

And, conversely, if either of these relations holds, the points A, X, B, Y form an H. R.

THEOREM (2)—*If* X, Y *are harmonic conjugates with respect to* A, B; *and* M *is the mid point of* AB; *then* $MA^2 = MX \,.\, MY = MB^2$; *and conversely.*



Since $AX : BX = AY : BY$,

$$\therefore \quad AX + BX : AX - BX = AY + BY : AY - BY,$$

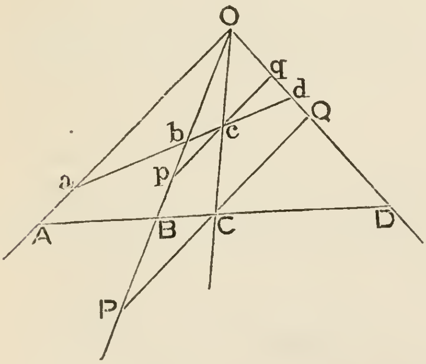
or $2\,MA : 2\,MX = 2\,MY : 2\,MA$,

whence $MA^2 = MX \,.\, MY.$

The converse follows by simply retracing the preceding steps.

Cor. X and Y move in opposite directions.

THEOREM (3)—*If the points of section of a pencil of four rays by any one transversal form an* H. R., *then the points of section of the pencil by every transversal will form an* H. R.



Let pencil, vertex O, be cut harmonically by a transversal in A, B, C, D.

Thro. C draw PCQ, ∥ to AO, to meet OB, OD in P, Q.

Then $AB : BC = AO : CP$,

and $AD : DC = AO : CQ$,

$\therefore \quad CP = CQ.$

Now let **abcd** be any other transversal to the pencil; and **pcq** ∥ to PCQ; where similar letters are on same ray.

$$\text{Then} \quad cp = cq.$$
$$\text{But} \quad ab : bc = aO : cp,$$
$$\text{and} \quad ad : dc = aO : cq.$$
$$\therefore \quad ab : bc = ad : dc.$$
$$\therefore \quad a, b, c, d \text{ is an H.R.}$$

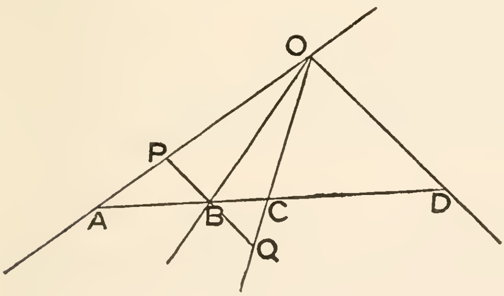
Def. A pencil through two pairs of harmonic conjugates is called a **harmonic pencil.**

Cor. (1). The intercepts made by three adjacent rays of a harmonic pencil, on a parallel to the fourth ray, are equal.

Cor. (2). O (ABCD) is harmonic if a transversal parallel to a ray is cut into equal segments by the remaining three.

THEOREM (4)—*The arms of an angle, and its internal and external bisectors form a harmonic pencil; and conversely, if in a harmonic pencil the angle between a pair of rays is right, then these rays are the internal and external bisectors of the angle between the other two.*

The first part follows at once from vi. 3, in connection with the *def.* of an H. R.



For the converse, let O (ABCD) be a harmonic pencil, such that $B\hat{O}D$ is right.

Draw PBQ ‖ to OD, meeting OA, OC in P, Q.

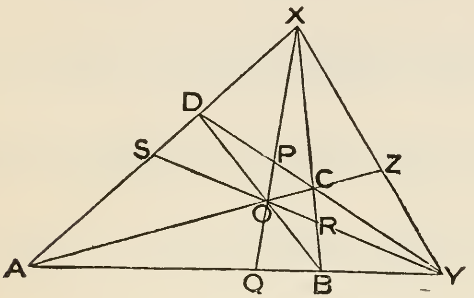
Then since PB = QB, and PBQ is ⊥ to OB,

$$\therefore \quad P\hat{O}B = Q\hat{O}B,$$

i.e. OB is internal bisector of $A\hat{O}C$.

Also OD, being ⊥ to OB, is external bisector of $A\hat{O}C$.

THEOREM (5)—*In a complete quadrilateral, if the intersections of the three diagonals are joined, and the joins produced; then all ranges and pencils are harmonic.*



In quad. ABCD, let
BC, AD meet in X,
AB, DC „ Y,
AC, BD „ O.

Also let XO cut CD in P, and AB in Q;
let YO cut BC in R, and AD in S;
and let XY cut AOC in Z.

Then, since O is a pt. within $\triangle$ ABX,

$\therefore$ (AQ : QB) (BC : CX) (XD : DA) = 1.

And, since YCD is a transversal of $\triangle$ ABX,

$\therefore$ (AY : YB) (BC : CX) (XD : DA) = 1.

$\therefore$ AQ : QB = AY : YB;

i.e. (AQBY) is harmonic.

$\therefore$ X (AQBY) and O (AQBY) are harmonic.

And $\therefore$ so also are X (AOCZ), Y (AOCZ), Y (ASDX), &c.

Note—The preceding Theorem suggests a mode of drawing the fourth ray of a harmonic pencil, when three consecutive rays are given. For let XA, XQ, XB be three given rays; take O any pt. in XQ, and let BO, AO meet XA, XB in D, C; let DC meet AB in Y; then XY is the fourth harmonic ray.

Cor. The Theorem is also true for the more extended definition of a complete quadrilateral given in the *Note* on p. 299.

EXERCISES ON HARMONIC RANGES.

1. The join of the points of contact of two sides of a triangle with its in-circle, meets the third side at the harmonic conjugate of the third point of contact, with respect to the two corners in that side.

NOTE—*Use Menelaus' Theorem.*

2. A similar Theorem to the last holds for each of the ex-circles.

3. Through any point in an altitude of a triangle lines are drawn from the ends of the base; if the points in which these lines meet the opposite sides are joined to the foot of the altitude, the joins make equal angles with the altitude.

NOTE—iii. *Addenda* (19) *is a particular case of this.*

4. From a fixed point two variable transversals are drawn to two fixed intersecting lines; if the points of section are joined transversely, find the Locus of the intersection of the joins.

5. If a line is drawn across a pair of orthogonal circles, it is harmonically divided by the circumferences if it goes through the centre of either.

6. Conversely to the last Exercise—If a circle is drawn through a pair of harmonic conjugates with respect to the ends of a diameter of another circle, then the circles cut orthogonally.

7. If a line touches two circles, then any circle co-axal with them cuts it in points which are harmonic conjugates with respect to the points of contact.

8. Any point on the circumference of a circle is joined to the ends of a chord; show that the joins (produced if necessary) cut the diameter perpendicular to the chord in points which are harmonic conjugates with respect to the ends of that diameter.

9. If X, Y are a pair of harmonic conjugates with respect to the ends A, B of a diameter of a circle, and P is any point in the perpendicular to AY at Y; then PX is cut harmonically by the circle.

10. If a transversal is drawn to a triangle, so as to bisect one of its sides, then the parallel to the bisected side, through the opposite corner, meets the transversal in a point which forms a harmonic range with the three points in which the transversal cuts the sides.

11. Two circles cut in A, B; if XX′ is any diameter of one, and YY′ any diameter of the other; and if XY, X′Y′ meet in Z, and XY′, X′Y in Z′; then the circle on ZZ′ as diameter goes through A, B.

NOTE—*Use Theorems* (5), (4) *and Menelaus'.*

SECTION viii—POLES AND POLARS.

Def. That line through the inverse of any point, with respect to a circle, which is perpendicular to the diameter containing the point, is called the **polar** of the point; and, conversely, the inverse of the foot of the perpendicular from the centre of a circle on any line is termed the **pole** of the line with respect to the circle—the point and its inverse being taken, in each case, on the same side of the centre.

Thus if X, Y are inverse points; then X, and the perpendicular to XY through Y; or Y, and the perpendicular to XY through X; are respectively pole and polar of each other with respect to the circle of inversion.

Note (1)—From the definition of pole and polar it follows that—

1°, they lie on the same side of the centre;

2°, as one approaches the centre, the other recedes from it; and conversely;

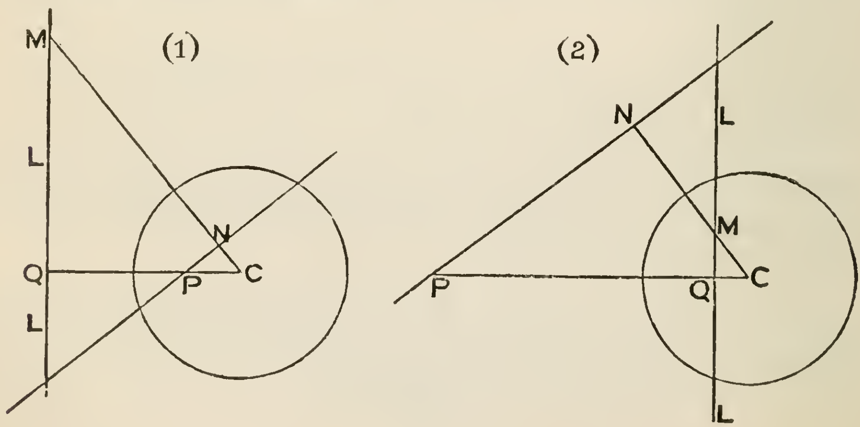
3°, in the case of a line touching a circle, the point of contact and the tangent are pole and polar of each other;

4°, the point of intersection, and chord of contact, of two tangents, are pole and polar.

5°, the angle subtended at the centre of a circle by the join of two poles, is equal or supplementary to the angle between their polars.

Note (2)—As pole and polar have been defined only *with respect to* a circle of inversion, it is unnecessary to make further explicit mention of this circle, but its existence is always to be tacitly understood.

THEOREM (1)—*When a line goes through a fixed point, its pole lies on the polar of that point; and, conversely, when a point lies on a fixed line its polar goes through the pole of that line.*



Let P, L be fixed pt. and line, pole and polar of each other; C centre of ⊙ of inversion; Q the pt. where CP meets L.

1°, let CN, ⊥ to any line through P, meet L in M.

Then PNMQ is cyclic.

$$\therefore \quad CN \,.\, CM = CP \,.\, CQ = (\text{rad.})^2.$$

$\therefore$ M is pole of PN.

2°, let M lie on L; and let PN be ⊥ to CM.

Then, as before, $CN \,.\, CM = (\text{rad.})^2$.

$\therefore$ PN is polar of M.

Cor. (1). The join of any two points is the polar of the intersection of their polars; and the intersection of two lines is the pole of the join of their poles.

Cor. (2). If a triangle has two of its corners, and their opposite sides, respectively pole and polar, then the third corner and side are also pole and polar.

Def. If two triangles are so related that each corner of one is the pole of a side of the other, they are said to be **polar triangles** with respect to each other.

Def. A triangle such that each of its corners and the opposite side are pole and polar is called **self-conjugate**.

THEOREM (2)—*In a self-conjugate triangle the ortho-centre is the centre of inversion.*

For, see fig. of iii. *Addenda* (19),

$$OA \,.\, OX = OB \,.\, OY = OC \,.\, OZ,$$

$\therefore$ ⊙ with O as centre, and radius whose square = any one of these rects, is ⊙ of inversion.

Note—From the nature of a pole and its polar, it appears that if a triangle is self-conjugate, its ortho-centre lies outside it; that is the triangle must be *obtuse-angled.*

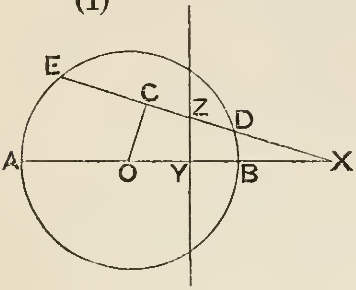
THEOREM (3)—*The three circles on the sides of a self-conjugate triangle as diameters cut the circle of inversion orthogonally.*

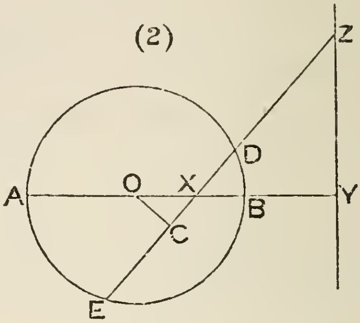
For (same fig. as in last Theorem) ⊙ on AB as diam. goes thro. X, Y.

$\therefore$ (tang. from O to it)^2 = OA . OX = (rad. ⊙ of inversion)^2.

$\therefore$ ⊙ on AB as diam. cuts ⊙ of inversion orthogonally.

THEOREM (4)—*If a line cuts a circle, and any point in the line is taken as a pole; then the point in which the polar cuts the line is the harmonic conjugate of the pole with respect to the two points in which the line meets the circle.*





Let E, D be pts. in which a line cuts ⊙, centre O; X a pt. in it taken as pole; Z the pt. in which the polar of X cuts ED; XBA the diam. thro. X, cutting the polar in Y; OC ⊥ to XE.

$$\text{Then } OX \, . \, OY = R^2;$$

$$\therefore \; OX^2 \mp OX \, . \, XY = CO^2 + CD^2; \quad \left[\begin{matrix} - \text{ in fig. (1),} \\ + \quad ,, \quad (2). \end{matrix}\right]$$

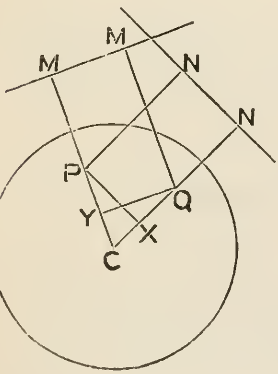
$$\therefore \; CO^2 + CX^2 \mp CX \, . \, XZ = CO^2 + CD^2;$$

$$\therefore \; CX \, . \, CZ = CD^2.$$

But C is mid pt. of ED.

∴ X, Z are harmonic conjugates to E, D.

THEOREM (5)—*(Salmon's) The distances of two points from the centre of a circle, have the same ratio as their distances each from the polar of the other with respect to the circle.*



Let P, Q be pts. whose respective polars, with regard to ⊙ C, are M, N.
Draw PN, QM ⊥s on N, M;
and PX, QY ⊥s to CQN, CPM.

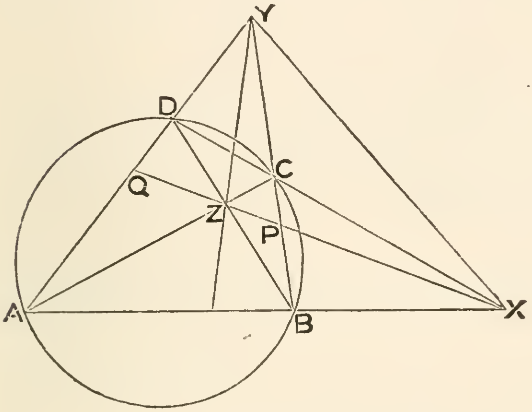
Then $CP \cdot CM = R^2 = CQ \cdot CN$.

And $CP \cdot CY = CQ \cdot CX$, $\because$ P, Y, X, Q are concyclic.

$\therefore$, by subtraction, $CP \cdot YM = CQ \cdot XN$;

$\therefore$ $PC : QC = XN : YM = PN : QM$.

THEOREM (6)—ABCD *is a cyclic quadrilateral; if* AB, DC *meet in* X; BC, AD *in* Y; *and* AC, BD *in* Z; *then* XYZ *is a self-conjugate triangle.*



Let XZ cut YCB in P, and YDA in Q.

Then (BPCY) and (AQDY) are harmonic ranges.

But the polar of Y cuts YCB in the harmonic conjugate of Y with respect to C, B; and cuts YDA in the harmonic conjugate of Y with respect to D, A.

$\therefore$ P, Q are the pts. in which the polar of Y cuts BC, AD:

i. e. XZ is the polar of Y.

Simrly. YZ is the polar of X.

$\therefore$ also XY is the polar of Z.

i. e. $\triangle$ XYZ is self-conjugate.

Cor. If secants XBA, XCD are drawn to a $\odot$; and if BC, AD meet in Y; and AC, BD in Z; then YZ is the polar of X, and YX is the polar of Z.

Note—The preceding Corollary gives a means of drawing a tangent to a given circle by a ruler only. For, taking a point X, outside the circle, and drawing secants &c., as in the Corollary; then since YZ is the polar of X, it cuts the circle in the points of contact of tangents from X; and joining these points to X, gives the tangents. Now all these lines are merely joins of points, and therefore can be drawn by the ruler only.

EXERCISES ON POLES AND POLARS.

1. If four points are collinear, their polars form a pencil; and the cross-ratio of this pencil is equal to that of the points.

2. The polar of a fixed point, with respect to any one of a co-axal system of circles, goes through a fixed point.

3. If a variable chord of a circle subtends a right angle at a fixed point, the Locus of its pole is a circle.

NOTE—*The pole is the inverse of the mid pt. of the chd.*

4. If at the corners of a cyclic quadrilateral tangents are drawn to its circum-circle, forming another quadrilateral; then—

1°, the four internal diagonals are concurrent, and form a harmonic pencil;

2°, the third diagonals are coincident.

NOTE—*Use Theorem* (6) *Cor.*

5. If one side of a cyclic quadrilateral is fixed, the join of the intersection of its diagonals with that of the sides adjacent to the fixed side, goes through a fixed point.

6. If a quadrilateral circumscribes a circle, the intersection of each pair of the three diagonals is the pole of the remaining diagonal.

7. Two triangles which are polar with respect to each other are in perspective.

8. If any number of circles have a pole and polar in common they are a co-axal system.

9. Given the position of a pair of opposite sides of a cyclic quadrilateral, and also the intersection of its diagonals, find the Locus of the centre of its circum-circle.

10. From a fixed point outside a fixed circle two variable lines are drawn to meet the circle in P, p and Q, q respectively: show that if PQ always goes through a fixed point, then pq always goes through another fixed point.

11. If a variable line is drawn through a fixed point in the line of centres of two fixed circles, the join of its poles, with respect to the circles, goes through a fixed point. When will the two fixed points coincide?

12. Find the Locus of a point such that its polars, with regard to three given circles, are concurrent.

NOTE—*The Locus is ⊙ orthogonal to the given ⊙s; and the pt. of concurrence is the other end of its diam. thro. variable pt.*

13. Two fixed lines meet on the circumference of a fixed circle: a variable point P is taken in one line; and its polar, with respect to the circle, meets the other in Q: show that PQ goes through a fixed point.

14. If two of the vertices of the self-conjugate triangle XYZ [Theorem (6)] lie on circles concentric with the circle round the quadrilateral; then will the third vertex lie on a concentric circle.

15. If XX′, YY′, ZZ′ are the diagonals of a quadrilateral circumscribing a circle, centre O; and L, M, N their mid points; then the ratios of OX . OX′, OY . OY′, OZ . OZ′, to each other, are respectively the same as those of OL, OM, ON. (W. S. McCay, *Educational Times. Reprint*, Vol. XXXIX. p. 88.)

16. Show that, assuming *Pascal's Theorem* [Section ii. Theorem (6) *Note*], the following Theorem (*Brianchon's*) can be deduced by Polars—The joins of the opposite corners of a hexagon which circumscribes a circle, are concurrent.

Def. The circle with respect to which a triangle is self-conjugate is called *the polar circle* of the triangle.

17. Using the *extended* definition of a complete quadrilateral given in the *Note* on p. 299; if XBA, XCD, YCB, YDA are the four lines; L, M, N the mid points of diagonals AC, BD, XY; P, Q, R the points where the diagonals intersect; O_1, O_2, O_3, O_4 the ortho-centres of the four triangles XBC, XAD, YAB, YCD; and Ω the centre of the circum-circle of PQR; then—

1°, the five points O_1, O_2, O_3, O_4, Ω are collinear;

2°, this line of collinearity is perpendicular to LMN, and is the radical axis of the three circles whose diameters are the diagonal;

3°, the polar circles of the above four triangles, and the circle Ω, have LMN as their radical axis;

4°, the set of three circles and the set of five circles cut orthogonally.

NOTE—*Unless* $\widehat{XAY}$ *is obtuse,* 2 *of the* Δ^s *will not be obtuse-angled, and* ∴ *will not have polar circles. The proofs of the theorems depend on properties given on* pp. 299, 352 (*Ex.* 7), 368, 370, 373.

18. Show that the harmonic section of a line by a circle, pole, and polar, may be proved by Inversion.

NOTE—*In figs. of Theor.* (4) *take* X *as centre of inversion; and for the radius of inversion take in fig.* (1) *the tangent from* X, *and in fig.* (2) *the semi-chd. thro.* X ⊥ *to* AB. *Then the* ⊙ *inverts into itself;* O, Y *are inverse pts., and* C, Z *are inverse pts. It will readily follow that*

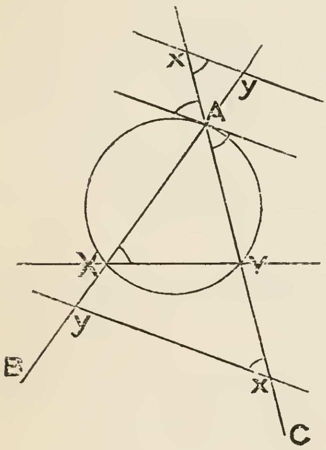
$$2\,XD \cdot XE = XZ\,(XD + XE).$$

SECTION ix—THE MODERN GEOMETRY OF THE TRIANGLE.

Def′— Triangles, in the same plane, are said to be **inversely similar**, when they are equiangular, but cannot have their sides placed parallel without first turning one of them completely over; as, for example, by rotating it about some line in that plane, as a hinge, until it is again in the original plane.

Note—If we wish to emphasize that the similarity of Δ^s is *not* inverse, we shall call them *directly similar*.

Def′—If two transversals are drawn across an angle, so as to make with its arms (or arms produced) two *inversely similar* triangles, then these transversals are said to be **anti-parallel** to each other with respect to the angle.



Thus in the diagram if XY, **xy** are transversals to BAC; so that X, **y** are in AB; Y, **x** in AC; and $\widehat{AXY} = \widehat{A\mathbf{xy}}$; i.e. so that Δ^s AXY, A**xy** are inversely similar; then XY, **xy** are anti-$\|^s$ to each other with respect to $\widehat{BAC}$.

Note (1)—Of course it follows that X, **y**, **x**, Y are concyclic; that the tangent at A to the circum-$\odot$ of Δ AXY is $\|$ to **xy**; and that, if S is the circum-centre of Δ AXY, SA is $\perp$ to **xy**.

Note (2)—Obviously all anti-$\|^s$ to the same line are $\|$ to each other.

Note (3)—It is also obvious that X, Y are respectively inverse points to **y**, **x**, with respect to A as centre of inversion; and, conversely, that if two systems of points are inverse, the one to the other, then the join of any pair of points in the one system is anti-$\|$ to the join of the corresponding pair in the other system; the centre of inversion being the vertex of the angle with respect to which the joins are anti-$\|$.

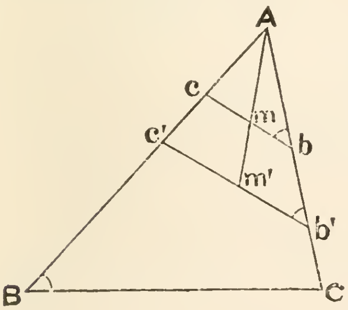
Note (4)—It will be at once apparent, by considering a diagram, that anti-$\|^s$ to two sides of a $\triangle$ are equally inclined to the third side: also it will be easily seen that if the join of their extremities, not in the third side, is $\|$ to the third side, these anti-$\|^s$ are equal; and, conversely, that, if they are equal, the above join is $\|$ to the third side; for, in each case, the points in which the anti-$\|^s$ meet the third side are the corners of a symmetrical trapezium (cf. p. 160): hence also, in each case, these points are concyclic; and the centre of the $\odot$ thro′ them lies in the bisector of the $\wedge$ between the anti-$\|^s$.

Note (5)—The analogy between $\|^s$ and anti-$\|^s$ appears from considering that, if two transversals are drawn across an $\wedge$, so as to make with its arms (or arms produced) two *directly similar* $\triangle^s$, then these transversals are $\|$.

Note (6)—X y, x Y are anti-$\|^s$ with respect to the $\wedge$ between XY and xy: hence it will be seen that if a quad′ is cyclic, either opposite pair of its sides are anti-$\|$ with respect to the $\wedge$ between the other pair; and conversely.

The Student should make himself familiar with the above facts: they are of much use in what follows.

THEOREM (1)—*The locus of the mid points of anti-parallels to a side of a triangle is a straight line.*



In $\triangle$ ABC, let bc, b′c′ be any anti-$\|^s$ to BC; so that $A\hat{b'}c' = A\hat{B}C = A\hat{b}c$.

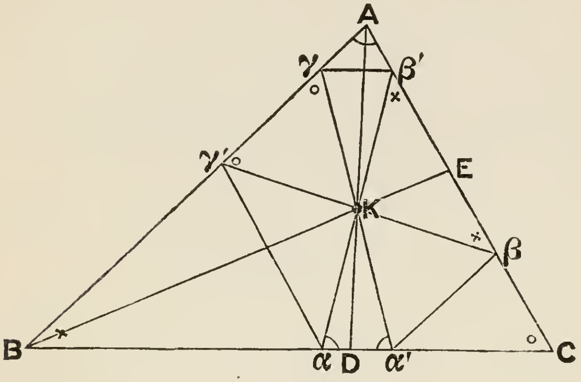
$\therefore$ bc, b′c′ are $\|$.

$\therefore$, if m′ is mid p′t of b′c′, Am′ is median of $\triangle$ A b′c′; and $\therefore$, by i. Addenda (12) Cor′, bisects all lines such as bc which are $\|$ to b′c′:

i.e. the Locus of the mid pt′s of all anti-$\|^s$ to BC is a st′ line.

Def′—As the Locus of the mid points of all parallels to a side of a triangle is called the *median* of the triangle with respect to that side, so the Locus of the mid points of all anti-parallels to a side of a triangle is called the symmedian of the triangle with respect to that side; and as there are three medians so there are three symmedians.

THEOREM (2)—*The three symmedians of a triangle are concurrent.*



Let two of the symmedians AD, BE (of $\triangle$ ABC) cut in K.

Draw $\alpha K \beta'$, $\beta K \gamma'$, $\gamma K \alpha'$, respectively anti-$\|^s$ to AB, BC, CA, so that

α, α' are in BC; β, β' are in CA; and γ, γ' are in AB.

Then, by def′ of anti-$\|^s$, the $\wedge^s$ sim′ly marked in the fig′ are equal.

Also AD bisects $\beta\gamma'$, and BE bisects $\gamma\alpha'$.

$\therefore \quad K\alpha = K\alpha' = K\gamma = K\gamma' = K\beta = K\beta'$.

$\therefore$ K is on the symmedian from C :

i.e. the three symmedians have a common point K.

Cor′—α, α', β, β', γ, γ' are concyclic; and K is the centre of the $\odot$ round them. Hence also $\alpha\alpha'\beta'\gamma$, $\beta\beta'\gamma'\alpha$, $\gamma\gamma'\alpha'\beta$ are each rectangles.

Def′—The point of concurrence of the symmedians of a triangle is called its **Lemoine** (or **symmedian**) **point.**

Note (1)—Since the sides of a $\triangle$ and of its pedal $\triangle$ are evidently anti-$\|^s$, the sides of the pedal $\triangle$ are bisected by the symmedians: hence we get the following simple construction for the Lemoine point. Join the vertex of each angle of a $\triangle$ to the mid point of that side of its pedal $\triangle$ which meets the arms of the angle: the three joins produced cointersect in the Lemoine point.

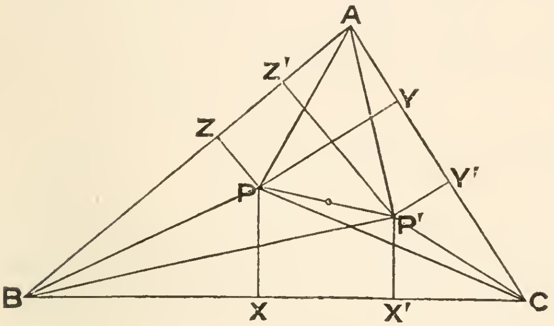
Note (2)—Excepting the centroid, the Lemoine point is the most important in the Geometry of the $\triangle$.

Def′—The circle through the six points in which anti-parallels to the sides of a triangle, through its Lemoine point, meet the sides, is called the **second Lemoine circle** of that triangle, from the name of its discoverer.

Note—This $\odot$ is also called the *cosine-circle* of the $\triangle$, $\because$ the intercept it makes on BC $= 2K\alpha\cos A$; so that the intercepts it makes on the sides are proportional to $\cos A$, $\cos B$, $\cos C$.

Def′—If from the vertex of an angle two lines are drawn making equal angles with its bisector, these lines are said to be **isogonal** with respect to that angle.

THEOREM (3)—*If three lines, drawn from the corners of a triangle, are concurrent, then the three lines isogonal to these are also concurrent.*



Let any p′t P be joined to the corners of a $\triangle$ ABC; and let P′ be the p′t such that

$$P'\widehat{A}C = P\widehat{A}B,$$

and

$$P'\widehat{B}C = P\widehat{B}A;$$

so that P′A, P′B are isogonal to PA, PB.

Join P′C; and drop PX, P′X′ $\perp^s$ on BC; PY, P′Y′ $\perp^s$ on CA; and PZ, P′Z′ $\perp^s$ on AB.

Then PZ : P′Y′ = PA : P′A, by sim′r $\triangle^s$ PAZ, P′AY′,

= PY : P′Z′, by sim′r $\triangle^s$ PAY, P′AZ′.

$\therefore$ PZ . P′Z′ = PY . P′Y′, and sim′ly = PX . P′X′.

$\therefore$ PX : PY = P′Y′ : P′X′.

Now let CQ be isogonal to CP: then, as above,

PX : $\perp$ from Q on CA = CP : CQ = PY : $\perp$ from Q on CB.

$\therefore$ $\perp$ from Q on CB : $\perp$ from Q on CA = PY : PX

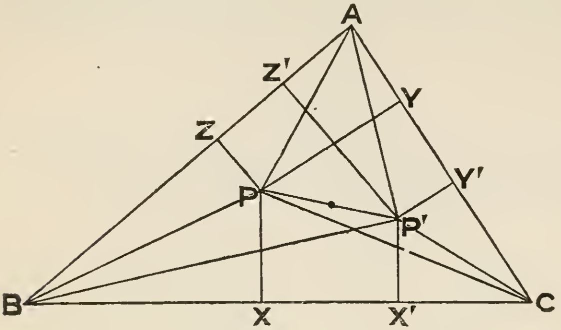
= P′X′ : P′Y′.

$\therefore$ Q is on CP′; or CP′ is isog′ to CP :

i.e. the isogonals to PA, PB, PC are concurrent.

Def′—If two points are so related to a triangle that the three joins of one of them to the corners of the triangle are isogonal to the three joins of the other, then the points are called **isogonal conjugates**.

Note—The analogous Theorem that—*If lines from the corners meet the opposite sides in collinear points, so do their isogonals*—is given as Exercise 7 at the end of this Section.



Note (1)—Hence it follows that the necessary and sufficient test that P, P′ may be isog′ conj′s is that

$$PX \, . \, P'X' = PY \, . \, P'Y' = PZ \, . \, P'Z'.$$

From this property P, P′ are sometimes also called *inverse points with respect to the* △ ; but the term *inverse points* has long been applied to those defined (as in Section vi) *with respect to a* ⊙ ; and so, if this term is used in the double sense, we must, to prevent ambiguity, add the phrase '*with respect to a* △,' or '*with respect to a* ⊙.' The term '*isogonal conjugates*' avoids the necessity for this addition; and is adopted by the *A. I. G. T.* in their Syllabus.

Cor′—The six projections of isog′ conj′s on the sides of a △ are concyclic; and the centre of the ⊙ round them is the mid p′t of the join of the isog′ conj′s.

For, by sim′r △s, we have

$$BZ : BX' = BP : BP' = BX : BZ',$$

$$\therefore \quad BX \, . \, BX' = BZ \, . \, BZ',$$

$$\therefore \quad X, X', Z', Z \text{ are concyclic;}$$

and, as the centre of the ⊙ round them lies in ⊥s to XX′, ZZ′, at their mid p′ts, it is the mid p′t of PP′.

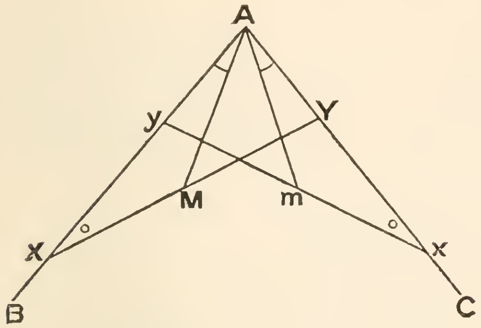
Sim′ly Y, Y′ are on the same ⊙.

Note (2)—It is easily seen that the circum-centre and ortho-centre of a △ are isog′ conj′s: hence the N. P. ⊙ is a particular case of this Cor′; and if, in the fig′ on p. 175, S is the circum-centre,

$$\text{then} \quad SD \, . \, OX = SE \, . \, OY = SF \, . \, OZ.$$

Note (3)—Isogonal conjugates may be outside the △ of reference: in particular if a point is on the circum-⊙, it may be easily seen that its isogonal conjugate is on two ||s, i.e. is at an infinitely removed distance ('Point at infinity').

THEOREM (4)—*The Lemoine point and centroid of a triangle are isogonal conjugates.*



If xy, XY are anti-$\|^s$ with respect to $B\hat{A}C$; then

m being mid p′t of xy,
and M ,, ,, XY;

A m is median of $\triangle$ A xy, and symmedian of $\triangle$ AXY ;

AM ,, ,, $\triangle$ AXY, ,, ,, ,, $\triangle$ Axy.

And if either of $\triangle^s$ AXY, Axy was turned over, and placed so that AX, Ax were in same direction, and AY, Ay in same direc′, then Am M would be in one line, $\because$ XY, xy would be $\|$, and M, m are their mid p′ts.

$$\therefore \quad X\hat{A}M = x\hat{A}m.$$

$\therefore$ AM, Am are isogonal with respect to $B\hat{A}C$.

$\therefore$ the point of concurrence of the three medians of a $\triangle$

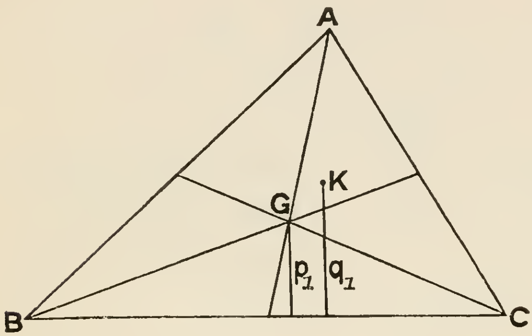
and ,, ,, ,, symmedians

are isogonal conjugates :

i.e. the Lemoine point and centroid are isogonal conjugates.

Cor′—$\perp^s$ from either of these p′ts on the sides of the $\triangle$ are inversely prop′l to $\perp^s$ from the other on the same sides.

THEOREM (5)—*The distances of the Lemoine point from the sides of a triangle are proportional to those sides.*



For if G is the centroid of $\triangle$ ABC; p_1, p_2, p_3 $\perp$s from G on BC, CA, AB, respectively; then,

$$\text{since } \triangle\,GBC = \triangle\,GCA = \triangle\,GAB,$$

$$\therefore\quad p_1 \cdot BC = p_2 \cdot CA = p_3 \cdot AB.$$

Now, if q_1, q_2, q_3 are $\perp$s from K (the Lemoine p′t) on BC, CA, AB; then, since G, K are isog′ conj′s, we have $p_1 \cdot q_1 = p_2 \cdot q_2 = p_3 \cdot q_3$.

$$\therefore\quad q_1 : BC = q_2 : CA = q_3 : AB.$$

i.e. K is the p′t whose dist′s from the sides are prop′l to those sides.

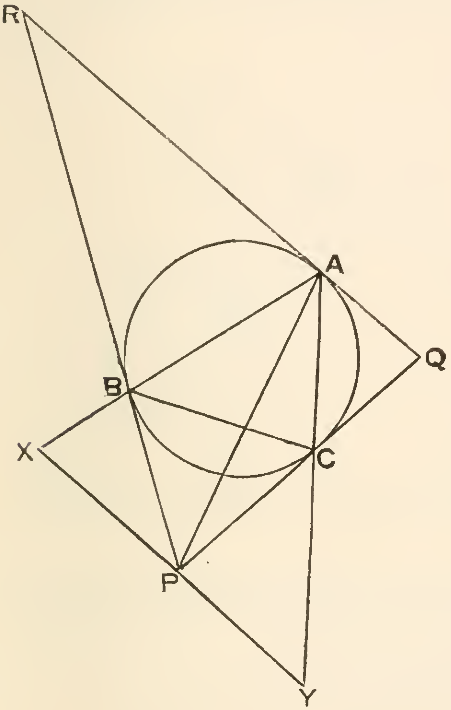
Cor′ (1)—From this easily follows *Grebe's* construction for K. Describe sq′s APQB, BUVC, CXYA on sides of $\triangle$ ABC (all externally, or all internally) and let QP, XY meet in α; PQ, VU in β; UV, YX in γ; then αA, βB, γC cointersect in K.

Cor′ (2)—If AK meets BC in D, then, as in fig. on p. 380

$$\begin{aligned} BD : CD &= \triangle\,AKB : \triangle\,AKC \\ &= q_3 \cdot AB : q_2 \cdot AC \\ &= AB^2 : AC^2 \\ \text{Also}\quad AK : KD &= \triangle\,AKB : \triangle\,KDB \\ \text{or } &= \triangle\,AKC : \triangle\,KDC \\ \therefore,\ \textit{addendo}, &= \triangle\,AKB + \triangle\,AKC : \triangle\,BKC \\ &= q_3 \cdot AB + q_2 \cdot AC : q_1 \cdot BC \\ &= AB^2 + AC^2 : BC^2 \end{aligned}$$

$\therefore$ K is the *mean centre* of A, B, C for mult′s prop′l to BC^2, CA^2, AB^2.

THEOREM (6)—*(Mathieu's) The Lemoine point of a triangle is the centre of perspective of that triangle and its polar triangle with respect to its circum-circle.*



Let tang's to the circum-⊙ of △ABC, at its corners, meet in P, Q, R; so that P is pole of BC, Q of CA, and R of AB.

Then PQR is polar △ of ABC with respect to its circum-⊙.

Draw the ∥ to QR thro' P, meeting AB in X, and AC in Y.

Then XPY is anti-∥ to BC. [*Note* (1), p. 378.]

Whence $\widehat{PXB} = \widehat{PBX}$,

and $\widehat{PYC} = \widehat{PCY}$.

∴ PX = PB = PC = PY.

∴ AP is a symmedian of △ ABC.

∴ the Lemoine p't of △ ABC is in AP.

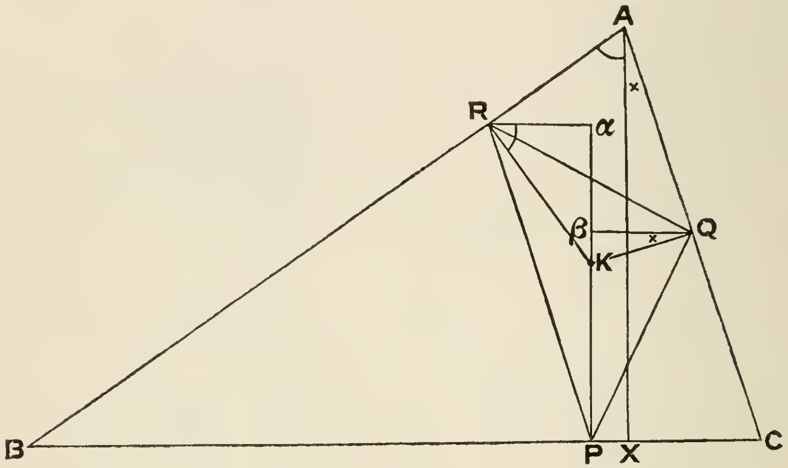
Sim'ly it is in BQ and CR:

i.e. it is the centre of perspective of △s ABC, PQR.

Cor'—Considering the ⊙ as the in- ⊙ of △ PQR, we see that the joins of the p'ts of contact of the in- ⊙ of a △ with its opposite corners concur; for they are the symmedians of the △ formed by joining the p'ts of contact. Cf. p. 336, Exercise 2. (3); and p. 376, Exercise 7.

This point of concurrence is sometimes known as *Gergonne's Point.* Hence the Lemoine point of a △ is the Gergonne point of its polar △ with respect to its circum-⊙.

THEOREM (7)—*(Catalan's) If from a point within a triangle perpendiculars are dropped on its sides; then, when the sum of the squares on these perpendiculars is* **minimum,** *the point is the centroid of the triangle formed by joining their feet; and is also the Lemoine point of the original triangle.*



Let K be the p′t in △ ABC for which, if KP, KQ, KR are ⊥ˢ on BC, CA, AB respectively,

$$KP^2 + KQ^2 + KR^2 \text{ is } min'.$$

Then, dotted letters denoting sim′r other p′ts,

$$\Sigma\, KP^2 < \Sigma\, K'P'^2,$$

$$\text{and } \therefore, \textit{à fortiori}, < \Sigma\, K'P^2;$$

∴ K is the *mean centre* of p′ts P, Q, R, for equal mult′s: p. 114, *Cor′* (1)

i.e. K is the centroid of △ PQR. p. 112, *Example* (1)

∴, if $R\alpha$, $Q\beta$ are ⊥ˢ on the median of △ PQR drawn from P,

then $R\alpha = Q\beta$, since △ PKR = △ PKQ;

$$\widehat{KR\alpha} = \text{comp't } \widehat{\alpha RA} = \widehat{BAX}, \text{ where AX is } \perp \text{ to BC};$$

$$\text{and } \widehat{KQ\beta} = \text{comp't } \widehat{\beta QA} = \widehat{CAX}.$$

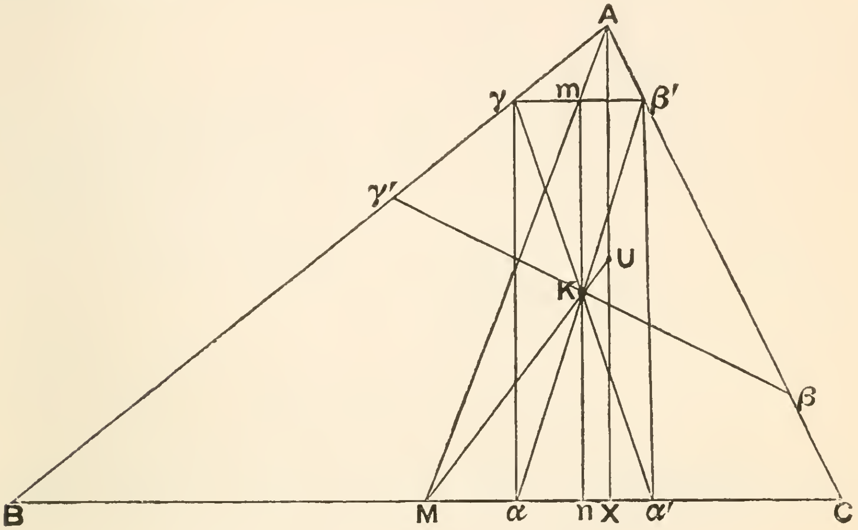
∴, from sim′r △ˢ $KR\alpha$, BAX; and sim′r △ˢ $KQ\beta$, CAX,

we get $KR : AB = R\alpha : AX = Q\beta : AX = KQ : AC$,

and, sim′ly $= KP : BC$:

∴ K is the Lemoine point of △ ABC.

THEOREM (8)—*(Schlömilch's) The three joins of the mid point of each side of a triangle, to the mid point of the corresponding altitude, cointersect in the Lemoine point.*



In $\triangle$ ABC let α, α' be p′ts in BC; β, β' in CA; γ, γ' in AB; such that $\alpha K\beta'$, $\beta K\gamma'$, $\gamma K\alpha'$ are the respective anti-$\|^s$ thro′ K (the Lemoine p′t) to AB, BC, CA.

$\therefore$ K is mid p′t of these lines; and $\alpha\,\alpha'\beta'\gamma$ is a rect′.

Then median AM (of $\triangle$ ABC) cuts $\beta'\gamma$ in its mid p′t m;

and if mK meets BC in n, Km = Kn;

and mKn is $\|$ to $\gamma\alpha$, and $\therefore$ to AX the alt′ from A.

$\therefore$ MK is median of $\triangle$ MAX;

and $\therefore$ bisects AX:

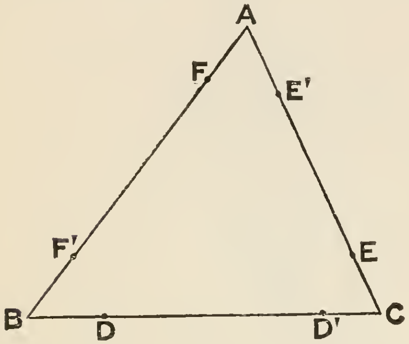
i.e. the join of the mid p′ts of BC, AX goes thro′ K.

Sim′ly for each other sim′r join.

Def′—As (for brevity) we have called the line joining two points the *join* of the points, so we shall call the point of intersection of two lines the cross of the lines.

Def′—Lines through the Lemoine point of a triangle, parallel to its sides, are called the **Lemoine parallels** of that triangle.

THEOREM (9)—*(R. F. Davis') If in each side (or side produced) of a triangle there is a pair of points, and the points are so related that each pair are concyclic with each other pair, then the* six *points are concyclic.*



For suppose that in Δ ABC, D, D′ in BC; E, E′ in CA; F, F′ in AB; are such that
D, D′, E, E′ are concyclic;
E, E′, F, F′ „ „
and F, F′, D, D′ „ „

Then AE . AE′ = AF . AF′;

$\therefore$ A is on the radical axis of $\odot^s$ DD′EE′, DD′FF′.

But the rad′ ax′ of these $\odot^s$ is their common ch′d DD′.

$\therefore$ the assumption of two separate $\odot^s$ leads to an impossibility.

$\therefore$ the six points are all on one $\odot$.

Cor′—If DE′, EF′, FD′ are respectively anti-|| to ED′, FE′, DF′, then the six p′ts D, D′, E, E′, F, F′ are concyclic; and conversely.

Note (1)—This is a Theorem of very extensive application: e.g. the fundamental property of the N. P. $\odot$ comes at once from it.

For in fig′ on p. 175, DE is || to $\alpha\beta$, and $\therefore$ anti-|| to XY.

Sim′ly EF is anti-|| to YZ, and FD to ZX.

$\therefore$, by above *Cor′*, D, E, F, X, Y, Z are concyclic.

i. e. $\odot$ round a pedal Δ bisects sides of original Δ.

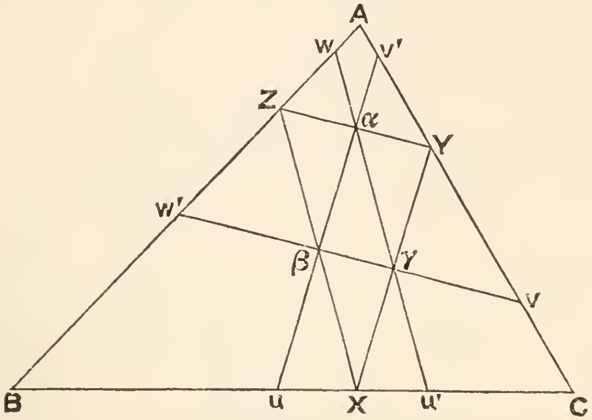
But XYZ is pedal Δ of Δ^s O$\alpha\beta$, O$\beta\gamma$, O$\gamma\alpha$;

and $\therefore$ $\odot$ round XYZ goes thro′ P, Q, R:

i. e. the *nine* p′ts D, E, F, X, Y, Z, P, Q, R are concyclic.

Note (2)—The Theorem is a converse of this—*If three circles are so situated that each circle cuts each of the other two, then their common chords are concurrent.* Cf. p. 348.

THEOREM (10)—*(H. M. Taylor's) If the joins of the mid points of the sides of a pedal triangle are produced to meet the sides of the original triangle, then the six points of meeting are concyclic; and the centre of the circle through them is the in-centre of the triangle formed by joining the mid points of the pedal triangle.*



Let XYZ be pedal △ of △ ABC; where X, Y, Z are in BC, CA, AB respectively; and let α, β, γ be the respective mid p'ts of YZ, ZX, XY

Let $\alpha\beta$ meet BC in **u**, $\beta\alpha$ meet CA in **v'**,
,, $\beta\gamma$,, CA ,, **v**, $\gamma\beta$,, AB ,, **w'**,
,, $\gamma\alpha$,, AB ,, **w**, $\alpha\gamma$,, BC ,, **u'**.

Then αw, αZ are equally inclined to AB, ∵ αw is ∥ to XZ.

∴ αw = αZ = αY, and sim'ly = αv'.

∴ wZYv' is cyclic.

∴ wv' is anti-∥ to YZ, and ∴ also to vw'. [Note (6), p. 379.]

Sim'ly uw' is anti-∥ to wu', and vu' to uv'.

∴, by the *Cor'* to *Davis' Theorem*, the six p'ts u, u', v, v', w, w', are concyclic.

Again uv', u'w, being ∥ to XY, XZ, are equally inclined to BC.

Also uv', u'w are anti-∥s to AB, AC.

∴ centre of ⊙ thro' u, u', v', w is in bisector of $\widehat{\beta\alpha\gamma}$. [Note (4), p. 379.]

Sim'ly ,, ,, v, v', w', u, ,, ,, $\widehat{\alpha\beta\gamma}$.

∴ centre of ⊙ thro' the six p'ts is the in-centre of △ $\alpha\beta\gamma$.

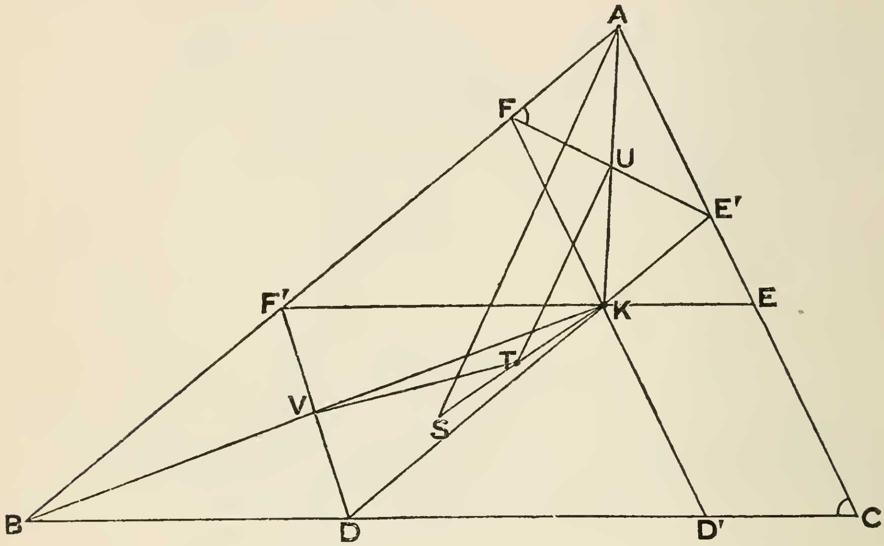
Cor'—These six p'ts are the projections of X, Y, Z on the sides of ABC; this easily appears from the fact that YαZ (e.g.) is diam' of ⊙ thro' w, v'.

THEOREM (11)—*If the Lemoine parallels of a triangle are drawn, then—*

1°, *the six points in which they meet the sides are concyclic;*

2°, *the centre of the circle round these points is the mid point of the join of the circum-centre and the Lemoine point;*

3°, *the intercepts made by this circle on the sides are in the triplicate ratio of these sides.*



Let DKE′, EKF′, FKD′ be the Lemoine ||ˢ of △ ABC;

so that D, D′ are in BC; E, E′ in CA; F, F′ in AB.

Then, since AFKE′ is a ▱,

∴ U (the cross of AK, FE′) is mid p′t of FE′.

But AUK is a symmedian of △ ABC.

∴ FE′ is anti-|| to BC, and ∴ also to EF′.

Sim′ly ED′ is anti-|| to DE′, and DF′ to FD′.

i.e., 1°, by the *Cor′* to *Davis' Theorem*, D, D′, E, E′, F, F′ are concylic.

Again, if S is the circum-centre of △ ABC,

then SA is ⊥ to FE′. [Note (1), p. 378.]

Now, if T is mid p′t of SK, TU is || to SA.

∴ TU is ⊥ to FE′ at U its mid p′t.

∴ centre of ⊙ thro′ F, E′ lies in TU.

Sim′ly same centre lies in TV, where V is mid p′t of DF′.

∴, 2°, T is centre of the ⊙ under discussion.

Lastly, if α, β, γ are $\perp^s$ from K; and $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are $\perp^s$ from A, B, C on BC, CA, AB respectively; then

$$\begin{aligned}
DD' : BC &= \alpha : \mathbf{p}_1, \\
CA : EE' &= \mathbf{p}_2 : \beta, \\
BC : CA &= \alpha : \beta. \\
\therefore\quad DD' : EE' &= (\alpha : \beta)(\alpha : \beta)(\mathbf{p}_2 : \mathbf{p}_1) \\
&= (BC : CA)(BC : CA)(BC : CA) \\
[\text{since } \mathbf{p}_1 . BC &= \mathbf{p}_2 . CA] \\
\text{i.e., } 3^\circ,\ DD' : EE' &= \text{triplicate of } BC : CA. \\
\text{Sim'ly } EE' : FF' &= \text{tripl' of } CA : AB.
\end{aligned}$$

Def'—The circle through the points in which the Lemoine parallels cut the sides of a triangle is called the **first Lemoine circle** of that triangle, from the name of its discoverer.

Note—From the property proved as 3° above it was called the *triplicate ratio circle* (T. R. ⊙) by Mr. Tucker, who (Q. J. xix. 76) investigated many of its properties.

$$\begin{aligned}
\textit{Cor'}\ (1)\text{—}BD : DD' &= (BD : DK)(DK : DD') \\
&= (BD : BF')(DK : DD') \\
&= (AB : BC)(AB : BC) \\
&= AB^2 : BC^2. \\
\text{Sim'ly } DD' : CD' &= BC^2 : AC^2. \\
\therefore\quad \text{also } BD : CD' &= AB^2 : AC^2. \\
\text{But } CD' : CE &= CA : CB, \text{ since } D'E \text{ is anti-}\parallel \text{ to } AB, \\
\therefore\quad BD : CE &= AB^2 : BC . CA.
\end{aligned}$$

Cor' (2)—Since, if $\triangle^s$ DEF, D'E'F' are drawn,

$$D\hat{F}E = D\hat{E'}E = B\hat{A}C = F'\hat{F}D' = F'\hat{E'}D',$$

and sim'r results,

$\therefore$ $\triangle^s$ FDE, E'F'D' are sim'r to $\triangle$ ABC.

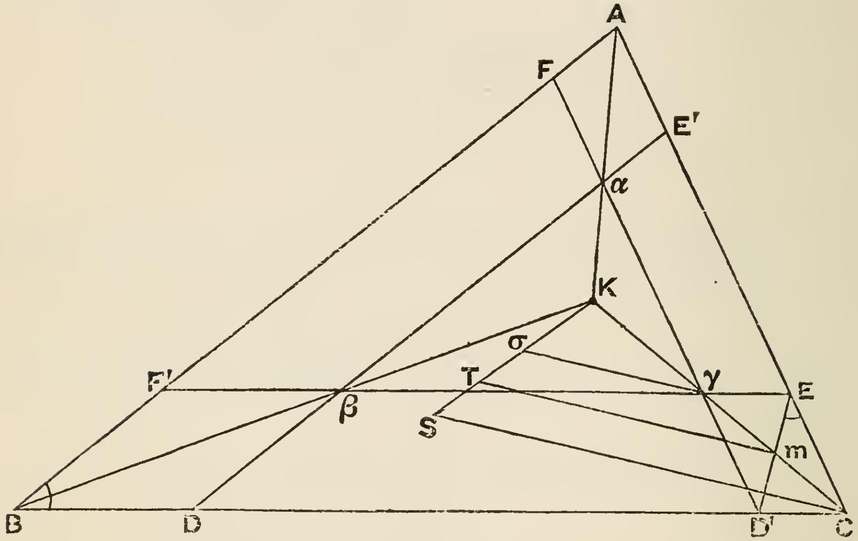
$\therefore$ also, since they are in the same ⊙,

$\therefore$ $\triangle$ FDE $\equiv$ $\triangle$ E'F'D'.

Cor' (3)—Since DF', FE' are anti-$\parallel^s$ to CA, CB; and DE' $\parallel$ to AB;

$\therefore$ DF' = FE', and sim'ly = ED'.

THEOREM (12)—*(Tucker's) If* K *is the Lemoine point of a triangle* ABC ; *and in* KA, KB, KC, *points* α, β, γ *are taken so that* $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ *are respectively parallel to* BC, CA, AB ; *and if the sides of the triangle* $\alpha\beta\gamma$ *are produced both ways to meet those of* ABC ; *then the six points of meeting are concyclic ; and the centre of the circle round them is the mid point of the join of the circum-centres of* ABC *and* $\alpha\beta\gamma$.



Let the p′ts of meeting be D, D′ in BC ; E, E′ in CA ; F, F′ in AB ; where E′$\alpha\beta$D, F′$\beta\gamma$E, D′$\gamma\alpha$F are each st′ lines.

Then, since CD′γE is a ▱, ∴ CγK bisects ED′.

∴ ED′ is anti-‖ to AB, and ∴ also to DE′.

Sim′ly FE′ is anti-‖ to EF′, and DF′ to FD′.

∴, by the *Cor′* to *Davis' Theorem*, D, D′, E, E′, F, F′ are concyclic.

Again, if S is circum-centre of △ ABC, and σ of △$\alpha\beta\gamma$; KσS is a st′ line, ∵ K is the centre of similarity of the △s.

Also SC is ⊥ to D′E ; [Note (1), p. 378]

and ∴ $\sigma\gamma$ (being ‖ to SC) is also ⊥ to D′E.

∴, if from m (mid p′t of D′E) a ‖ is drawn to SC, it will go thro′ T, the mid p′t of Sσ, and will contain the centre of the ⊙.

Sim′ly for other sim′ly drawn lines.

∴ T is the centre of the ⊙.

Def′—The circles got by varying the position of $\alpha\beta\gamma$ are called **Tucker's group (or family) of circles.**

Note—It is easily seen that the circum-⊙, 1st Lemoine ⊙, and 2nd Lemoine ⊙, are members of Tucker's family of ⊙s.

For in the case of the circum-⊙, α, β, γ are at A, B, C respectively;

„ „ 1st Lemoine ⊙, „ all at K;

and „ „ 2nd Lemoine ⊙, „ in AK, BK, CK produced.

It is to be noticed that when α, β, γ are outside △ ABC they are in the productions of AK, BK, CK, *not* of KA, KB, KC.

Cor′ (1)—If ED′, DF′, FE′ are produced both ways to form a △, its corners will be on the symmedians produced.

Cor′ (2)—By drawing a ‖ thro′ α to BC, it can be seen that

$$\mathrm{BD} : \mathrm{CD'} = \mathrm{PB} : \mathrm{PC}, \text{ where AK meets BC in P}$$

$$= \mathrm{AB}^2 : \mathrm{AC}^2 \quad [\textit{Theorem } (5)\ \textit{Cor}'\ (2)]$$

Also $\mathrm{CD'} : \mathrm{CE} = \mathrm{CA} : \mathrm{CB}$, since D′E is anti-‖ to AB.

$$\therefore\ \mathrm{BD} : \mathrm{CE} = \mathrm{AB}^2 : \mathrm{BC}\,.\,\mathrm{CA}.$$

Cor′ (3)—Since, if △s DFE, D′F′E′ are drawn,

$$\mathrm{D\hat{F}E} = \mathrm{D\hat{E}'E} = \mathrm{B\hat{A}C} = \mathrm{F'\hat{F}D} = \mathrm{F'\hat{E}D'},$$

with sim′r results;

∴ △s FDE, E′F′D′ are sim′r to △ ABC.

∴ also, since they are in the same ⊙,

△ FDE ≡ △ E′F′D′.

Cor′ (4)—That Taylor's ⊙ is also a Tucker ⊙ will appear thus—

In the fig′ of Theorem (10)

$$\mathrm{Bu'} : \mathrm{Bw} = \mathrm{AB} : \mathrm{BC},$$

$$\mathrm{Bw} : \mathrm{Cv'} = \mathrm{AB} : \mathrm{AC}.$$

$$\therefore\ \mathrm{Bu'} : \mathrm{Cv'} = \mathrm{AB}^2 : \mathrm{BC}\,.\,\mathrm{CA}.$$

$$\text{And } \mathrm{u'\hat{v}'w'} = \mathrm{u'\hat{w}w'} = \hat{\mathrm{C}} = \mathrm{v\hat{u}w},$$

with sim′r results.

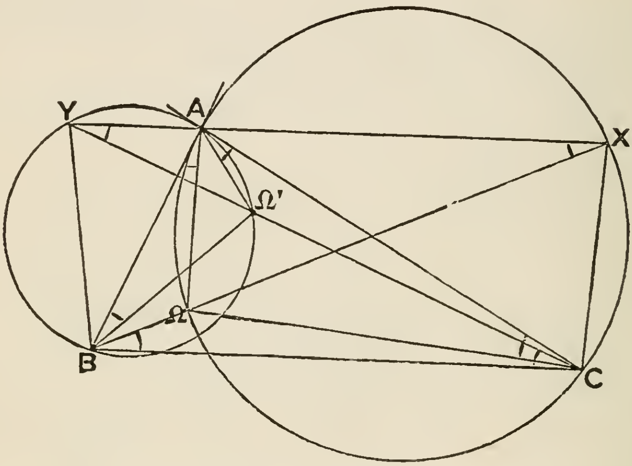
∴ △s u′w′v′, vwu are sim′r to △ ABC.

∴ also, since they are in the same ⊙,

△ u′w′v′ ≡ △ vwu.

Comparing these results with *Cor′s* (2) and (3) above, it is apparent that the Taylor ⊙ is a Tucker ⊙, u′w′v′ corresponding to FDE, in the two fig′s.

THEOREM (13)—*If a circle touches the side* AB *(of a triangle* ABC*) at* A, *and goes through* C ; *and a circle touches the side* AC *at* A, *and goes through* B ; *and if* XAY, *parallel to* BC, *has* X *on the circle through* C, *and* Y *on the circle through* B ; *and if* BX *cuts the first circle in* Ω, *and* CY *cuts the second circle in* Ω′ ; *then the six angles* ΩAB, ΩBC, ΩCA, Ω′AC, Ω′CB, Ω′BA, *are equal.*



For $\widehat{AXB}$ = each of $\Omega\widehat{AB}$, $\Omega\widehat{BC}$, $\Omega\widehat{CA}$.

And $\widehat{AYC}$ = each of $\Omega'\widehat{AC}$, $\Omega'\widehat{CB}$, $\Omega'\widehat{BA}$.

Whence $\widehat{A\Omega C} = \widehat{B} + \widehat{C}$ = supp′t of $\widehat{A}$;

and ∴ $\widehat{AXC} = \widehat{A}$.

Also $\widehat{A\Omega' B} = \widehat{B} + \widehat{C}$ = supp′t $\widehat{A}$.

and ∴ $\widehat{AYB} = \widehat{A}$.

∴ BCXY is a symmetrical trapezium (cf. p. 160)

and ∴ is cyclic.

∴ $\widehat{AXB}$ (which = $\widehat{XBC}$) = $\widehat{AYC}$.

∴ the six $\wedge^s$ under consideration are equal *.

* This construction is a modification of the one contributed to the Syllabus of the *A. I. G. T.* by Mr. R. F. Davis.

Cor' (1)—Ω, Ω' are isogonal conjugates.

Cor' (2)—$C\hat{\Omega}A$ = supp't of $\hat{A}$ = $A\hat{\Omega'}B$.

$A\hat{\Omega}B$ = supp't of $\hat{B}$ = $B\hat{\Omega'}C$.

$B\hat{\Omega}C$ = supp't of $\hat{C}$ = $C\hat{\Omega'}A$.

Cor' (3)—If the ⊙ touching BC at B, and going thro' A; and the ⊙ touching CA at C, and going thro' B; are drawn, they will cointersect in Ω with the ⊙ already drawn thro' C. Also the ⊙ touching AB at B, and going thro' C; and the circle touching BC at C, and going thro' A; will cointersect in Ω' with the ⊙ already drawn thro' A.

Def'—If Ω, Ω' are points within a triangle ABC, such that the angles ΩAB, ΩBC, ΩCA, Ω'AC, Ω'CB, Ω'BA are equal—where the corners of the triangle are lettered so that the cycle ABC is anti-clockwise—then Ω, from which the angles are measured round in an anti-clockwise direction, is called the **positive Brocard point** of the triangle; and its isogonal conjugate Ω', from which the angles are measured round in a clockwise direction, is called the **negative Brocard point** of the triangle.

Def'—The magnitude of the common value of these six angles is usually denoted by ω, and is called the **Brocard angle** of the triangle.

Note (1)—The terms *positive* and *negative*, as applied to the Brocard angles, are used in accordance with the universal custom in those parts of mathematics where angles are considered with respect to their sign as well as their magnitude.

Note (2)—Since △s XCA, ABC are sim'r, we have this graphical ruler construction for the Brocard ∧ of a △ ABC—*On* AC *describe externally the* △ AXC *sim'r to* CAB; *and join* XB. *Then* XBC *is the Brocard angle.* From this we see that, since the species of fig' XABC is fixed, when that of △ ABC is fixed, the magnitude of the Brocard ∧ is the same for all sim'r △s.

Note (3)—If the magnitude of $A\hat{B}C$ is fixed, then, by drawing BC always ∥ to itself, we can increase the Brocard ∧ until BC touches the ⊙, and then

$$B\hat{A}C = B\hat{C}A.$$

∴ if one ∧ of a △ is fixed, the Brocard ∧ is *max'* when the △ is isosceles, and ∴ is *max'* altogether when △ is equilat'; but then it is easily seen to be 30°.

Note (4)—In Theorem (11) since, by Cor' (3), ED', DF', FE' are equal, they subtend equal ∧s at the circumf' of the T. R. ⊙.

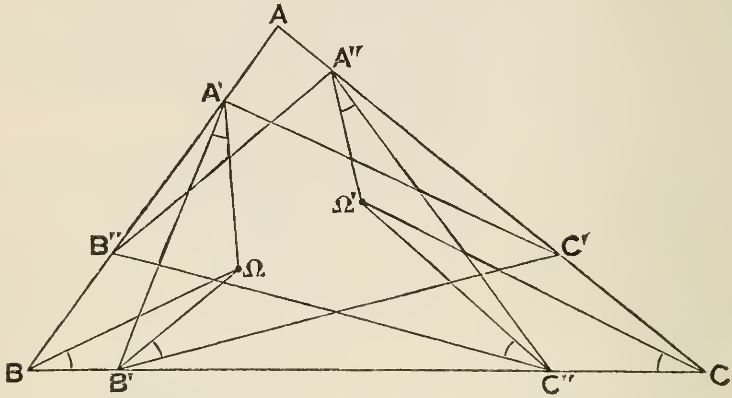
$$\therefore \quad K\hat{F}E = K\hat{E}D = K\hat{D}F = K\hat{F'}D' = K\hat{E'}F' = K\hat{D'}E'.$$

∴ K is the pos' Brocard p't of △ D'E'F'.

and ,, neg' ,, △ DEF.

Def'—The arcs AΩC, AΩ'B, BΩA, BΩ'C, CΩB, CΩ'A are called the **Brocard arcs** of the triangle.

THEOREM (14)—*If triangles* ABC, A′B′C′, A″B″C″ (*lettered in an anti-clockwise cycle*) *are directly similar; and* A′B′C′, A″B″C″ *are so inscribed in* ABC *that* A′ *is on* AB, *and* A″ *on* AC; *then will* ABC, A′B′C′ *have the same positive Brocard point*, *and* ABC, A″B″C″ *the same negative Brocard point.*



The Δ^s being sim′r have the same value for ω, the Brocard $\wedge$.

Let Ω be the pos′ Brocard p′t of Δ A′B′C′, and Ω' the neg′ Brocard p′t of Δ A″B″C″.

Make joins as in the fig′.

Then $A'\hat{\Omega}B'$ = supp′t $A'\hat{B'}C'$ = supp′t $A\hat{B}C$.

$\therefore$ Ω concyclic with A′, B, B′.

$\therefore$ $\Omega\hat{B}B' = \Omega\hat{A'}B' = \omega$.

Sim′ly for $\wedge^s$ at other corresponding corners.

$\therefore$ Ω is pos′ Brocard p′t of Δ ABC.

Again, $A''\hat{\Omega'}C''$ = supp′t $A''\hat{C''}B''$ = supp′t $A''\hat{C}C''$.

$\therefore$ Ω' concyclic with A″, C, C″.

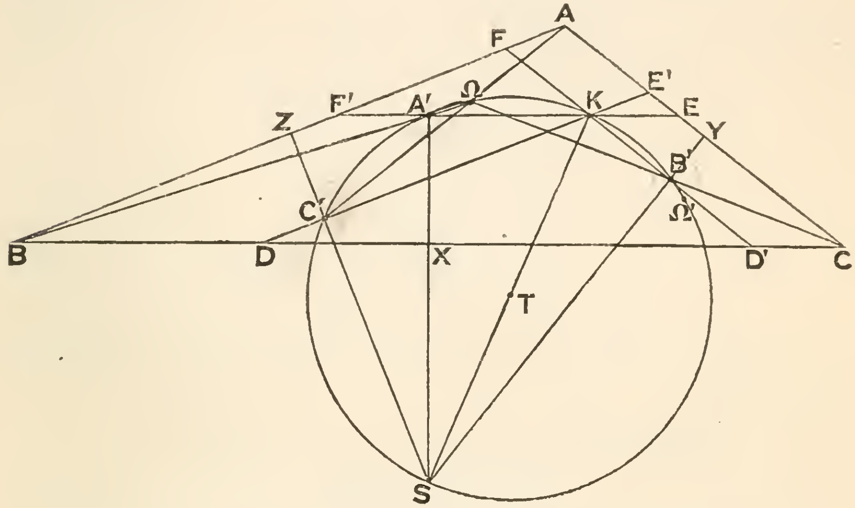
$\therefore$ $\Omega'\hat{C}C'' = \Omega'\hat{A''}C'' = \omega$.

Sim′ly for $\wedge^s$ at other sim′r corners.

$\therefore$ Ω' is neg′ Brocard p′t of Δ ABC.

Cor′—Ω, Ω' are respective centres of similarity in the two cases; and, as they are fixed p′ts, they are *permanent centres of similarity* for all positions of the inscribed Δ^s.

THEOREM (15)—*(Brocard's) If* S *is the circum-centre,* K *the Lemoine point, of a triangle* ABC ; *and if the circle on* SK *as diameter cuts the Lemoine parallels to* BC, CA, AB, *in* A′, B′, C′, *respectively ; then*—1°, AC′, BA′, CB′ *cointersect on the circle ; and their cross is the positive Brocard point ; and,* 2°, AB′, BC′, CA′ *cointersect on the circle ; and their cross is the negative Brocard point.*



Let EKF′, FKD′, DKE′ be the respective Lemoine ||s thro′ A′, B′, C′.

Then $\widehat{SA'K}$ is right, and ∴ A′S is ⊥ to BC, and ∴ meets it in its mid p′t X.

Sim′ly B′S, C′S meet sides CA, AB, in their mid p′ts Y, Z.

Let BA′, CB′ cut in Ω ; then, using *Theorem* (5)

A′X : BX = twice ⊥ from K on BC : BC

= ,, ,, CA : CA

= B′Y : CY.

∴ Δs A′BX, B′CY are sim′r.

∴ $\widehat{BA'X} = \widehat{CB'Y} = \widehat{SB'\Omega}$.

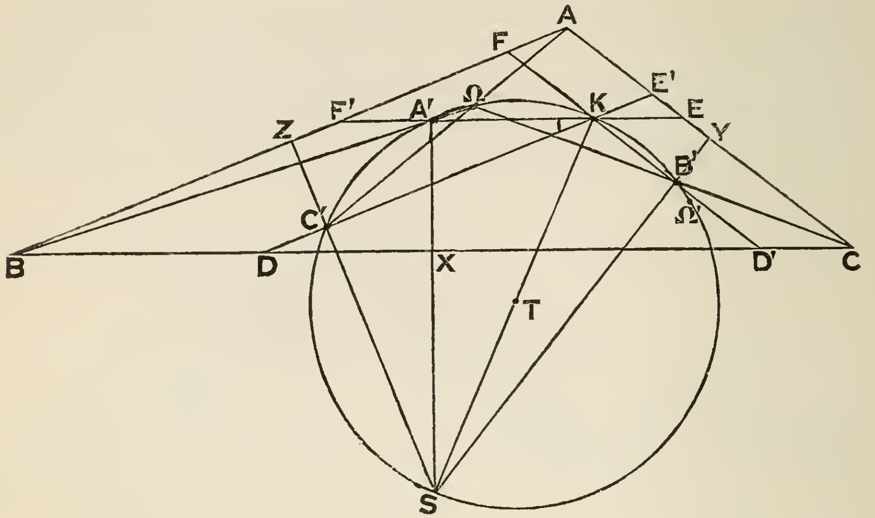
∴ Ω is concyclic with A′, B′, S, and ∴ also with C′.

i.e. BA′, CB′ cross on the ⊙ whose diam′ is SK.

Sim′ly BA′, AC′ can be shown to cross on the same ⊙ ;

and ∴, as BA′ cannot meet the ⊙ in three p′ts, they also cross in Ω :

i.e. AC′, BA′, CB′ cointersect in Ω on the ⊙ whose diam′ is SK.



In a precisely sim′r way it can be shown that AB′, BC′, CA′ cross in a p′t Ω′ on the ⊙.

$$\text{Again,}\quad \Omega\hat{B}C = \Omega\hat{C}A, \text{ from sim′r } \triangle^s\ A'BX,\ B'CY;$$

$$\text{and sim′ly} = \Omega\hat{A}B.$$

∴ Ω is the positive Brocard p′t.

Sim′ly Ω′ ,, negative ,,

So that the seven p′ts S, K, Ω, Ω′, A′, B′, C′, are concyclic; and the centre of the ⊙ round them is the same as that of Lemoine's first ⊙.

Def′—The circle on the join of the circum-centre and Lemoine point of a triangle as diameter is called the **Brocard circle** of the triangle, from the name of its discoverer.

Cor′ (1)—Since the 1st Lemoine ⊙ goes thro′ E, F′; and is concentric with the Brocard ⊙ thro′ K, A′;

∴ KE = A′F′.

∴ AA′, AK cut BC in p′ts equidistant from its mid p′t.

∴, by Exercise 2. (6) p. 336, AA′, BB′, CC′ concur; and their p′t of concurrence is the *isotomic conjugate* of K: i. e. △ A′B′C′ is in perspective with △ ABC.

Def′—The triangle A′B′C′, whose corners are the projections of S on the Lemoine parallels, is called **Brocard's first triangle**.

Cor' (2)—Since $A'\hat{C}'B'$ = supp't $A'\hat{K}B' = A\hat{C}B$,

$\therefore$ Brocard's 1st Δ is inversely sim'r to Δ ABC.

Cor' (3)—Since FA, KE' are equal and $\parallel$, $\therefore$ so also are FA, DC'; and $\therefore$ also AC', FD.

Hence sides of Δ FDE are $\parallel$ to ΩA, ΩB, ΩC, respectively.

Sim'ly ,, E'F'D' ,, Ω'B, Ω'C, Ω'A, ,,

Cor' (4)—Since $\Omega\hat{A}'K = \Omega\hat{B}C = \omega = \Omega'\hat{C}B = \Omega'\hat{A}'K$,

$\therefore$ Ω, Ω' are equidistant from K;

and $\therefore$ $\Omega\Omega'$ is $\perp$ to and bisected by SK.

Cor' (5)—$A'\hat{\Omega}C' = A'\hat{K}C' = A\hat{B}C$.

Cor' (6)—$\Omega\hat{T}\Omega' = 2\,\Omega\hat{T}K = 4\,\Omega\hat{A}'K = 4\omega$.

Cor' (7)—Since $F\hat{D}K = F\hat{A}C' = \omega$; and sim'r results;

$\therefore$ K is the neg' Brocard p't of Δ FDE.

Also, since $D'\hat{F}'K = D'\hat{C}A' = \omega$; &c.

$\therefore$ K is the pos' Brocard p't of Δ E'F'D'.

Cor' (8)—By Theorem (14) Δs FDE, ABC have same pos' Brocard p't; and Δs E'F'D', ABC have same neg' Brocard p't.

Cor' (9)—If AK, BK, CK meet the Brocard $\odot$ in A'', B'', C'', then A'' is the cross of the Brocard arcs AΩC, AΩ'B; B'' is the cross of BΩA, BΩ'C; and C'' is the cross of CΩB, CΩ'A.

For let Q be the cross of AΩC, AΩ'B; and let AQ meet BC in P. (Draw the fig' with Δ ABC acute-angled)

Then (noticing that Δs QAB, QCA are sim'r) it easily follows that

$$BP : CP = AB^2 : AC^2.$$

$\therefore$ Q is on the symmedian from A.

Also $B\hat{Q}C = 2\hat{A} = B\hat{S}C$,

$\therefore$ $S\hat{Q}B = S\hat{C}B$ = comp't $\hat{A}$,

$\therefore$ $S\hat{Q}K$ is r't;

i. e. Q is on the Brocard $\odot$.

Def'—The triangle A'' B'' C'', whose corners are the projections of S on the symmedians, is called **Brocard's second triangle**.

EXERCISES ON THE MODERN GEOMETRY OF THE TRIANGLE.

1. If PA, QA are isogonal with respect to an angle whose vertex is A, show that the join of the feet of the perpendiculars from P on the arms of the angle is anti-parallel to the join of the feet of the perpendiculars from Q on the same arms.

2. XAx, YAy are intersecting lines; AP, AQ are isogonal with respect to the angle XAY: if Xy is parallel to AQ, and Yx to AP, prove that XY, xy are anti-parallels.

3. At the vertex of each angle of a triangle are drawn a pair of isogonals; if three of these lines, each through a different vertex, are parallel, show that the other three cointersect on the circum-circle of the triangle.

4. From any point P perpendiculars PX, PY are dropped on the arms of an angle whose vertex is A; show that the perpendicular from A on XY is isogonal to AP, with respect to the angle A.

5. P, Q are isogonal conjugates with respect to a triangle ABC; and PX, PY, PZ are respectively perpendicular to BC, CA, AB; show that the rectangles PA . PX, PB . PY, PC . PZ, are in the same proportion as the lines QA, QB, QC.

6. Prove that the Lemoine point of a right-angled triangle bisects the altitude from the vertex of the right angle.

7. If three lines, drawn from the corners of a triangle, meet the opposite sides in collinear points, then the three lines isogonal to these also meet the opposite sides in collinear points. (*Royal University Scholarships*: 1889)

NOTE—*If* A′, B′, C′ *are the 1st set; and* A″, B″, C″ *the 2nd* (*where* B′, B″ *are in* AC *produced both ways*) *drop* $\perp^s$ *from* A, C *on* BB′ *and* BB″; *and use Menelaus' Theorem.*

8. From the Lemoine point of a triangle perpendiculars are dropped on its sides, and a second triangle formed by joining their feet: prove that the medians of the first triangle are respectively perpendicular to the sides of the second. (*Royal University Matriculation*: 1888.)

9. If P, Q are points in the side BC of a triangle ABC, such that AP, AQ are isogonal with respect to the angle BAC, prove that—

$$AP^2 : AQ^2 = PB \cdot PC : QB \cdot QC,$$
$$\text{and } AB^2 : AC^2 = BP \cdot BQ : CP \cdot CQ.$$

10. If ABC is a triangle, and bc anti-parallel to BC, so that corresponding letters denote equal angles; if the circum-circles of ABC, Abc cross in K, prove that

$$KB \cdot Kb : KC \cdot Kc = AB^2 : AC^2.$$

11. Two circles cut orthogonally; and PQ any diameter of the one cuts the common chord in L, and the other circle in O; if ALB is any chord of the first circle through L, prove that OP is a symmedian of the triangle APB.

NOTE.—Since the polar of PQ is [illegible] of AB with respect to [illegible] APB, [illegible] the Harmonic Theorem.

12. If PT touches, and PNT' cuts the same circle, show that triangles PNT, PTT' have a common Brocard angle, and a common Brocard point.

R. F. Davis.

13. Show that the join of any two points, and the join of their isogonal conjugates, subtend at any corner of the triangle of reference angles which are equal, or supplemental.

Gallatly.

14. If ABC is a triangle, CA is produced to E, and BA to F', so that AE, AF', BC are equal; AB is produced to F, and CB to D', so that BF, BD', CA are equal; BC is produced to D, and AC to E', so that CD, CE', AB are equal; show that D, D', E, E', F, F' are on a circle whose centre is the in-centre of ABC.

NOTE.—*This is* [illegible] *in a disguised form.*

15. Prove that the polars of the corners of a triangle, with respect to the corresponding opposite ex-circles, meet the external bisectors of its angles in six points which are concyclic; and that the circum-centre of the triangle formed by these polars is the ortho-centre of the original triangle.

Mackay—Educational Times, 1888.

16. If the symmedians AK, BK, CK, of a triangle ABC, meet its circum-circle in A', B', C' respectively, show that the triangle A'B'C' has the same Lemoine point and circles, and the same Brocard points and circle as the triangle ABC.

Simmons.

17. Show that if two triangles are co-symmedian (as, for instance, the triangles ABC, A'B'C' are, in the preceding Exercise), then the sides of the one are proportional to the medians of the other.

Simmons.

18. With the notation of Theorem (15) prove the following Theorems—

(1) $\Omega'A$, $\Omega'B$, $\Omega'C$ are perpendicular to the sides of the triangle formed by joining the feet of the perpendiculars from Ω on the sides of ABC.

(2) If $\Omega\Omega'$ is parallel to a side, the triangle ABC is isosceles.

(3) If $\Omega\Omega'$ goes through a corner of ABC, the opposite side is a mean proportional between the other sides.

(4) $K\Omega\Omega'$ is the Brocard angle of ABC.

(5) Joins of the mid points of sides of $A'B'C'$ with corresponding mid points of sides of ABC cointersect on mid point of $\Omega\Omega'$.

(6) An infinite number of triangles can be described having their corners on the circles $B\Omega C$, $C\Omega A$, $A\Omega B$; and their sides passing through A, B, C; such that they have the same Brocard angle, and positive Brocard point as ABC. (*Neuberg*)

(7) $(\Omega A : \Omega'A)\,(\Omega B : \Omega'B)\,(\Omega C : \Omega'C) = 1$.

(8) The Lemoine points of DEF, $D'E'F'$ are equidistant from T, and subtend at T an angle 2ω.

(9) One of the circles which touches the lines $D'E$, $E'F$, $F'D$ (produced as may be necessary) has the same centre as the Brocard circle, and the same radius as the N. P. circle of the triangle ABC.

(10) If $A\Omega$, $A\Omega'$ meet BC in X, X′ respectively,

$$BX : XC = AB^2 : BC^2$$

and $$BX' : X'C = BC^2 : CA^2.$$

(11) The *Lemoine line* (polar of K with respect to the circum-circle) is the radical axis of the Brocard circle and circum-circle.

(12) If $A\Omega$, $B\Omega$, $C\Omega$ are produced to meet the circum-circle again in α, β, γ, the triangle $\gamma\alpha\beta$ is identically equal to the triangle ABC, and has Ω for its negative Brocard point. (*R. F. Davis*)

(13) The joins of A, B, C to the mid points of $B'C'$, $C'A'$, $A'B'$, respectively, are concurrent.

19. Taking the figure of the Tucker circles, show that the following twelve angles are equal, viz.—the angles subtended at Ω by AF, BD, CE; the angles subtended at Ω' by AE', BF', CD'; and the angles which the sides of ABC make with those of DEF, $D'E'F'$, taken one way round for one triangle and the opposite way round for the other.

20. AX, BY, CZ are the altitudes of a triangle ABC; if K, K_1, K_2, K_3 are the respective Lemoine points of ABC, AYZ, BZX, CXY, prove that K is the mid point of the perpendiculars from K_1 on BC, from K_2 on CA, and from K_3 on AB. (*Wetzig: Educational Times*, LI)

21. If XYZ, DEF are respectively the pedal and medial triangles of a triangle ABC, prove that the *Simson lines* of XYZ with respect to any one of the points D, E, F, or of DEF with respect to any one of the points X, Y, Z, cointersect at the centre of the *Taylor circle* of ABC. (*Tucker*)

Def′—*The* **medial triangle** *of* ABC *is formed by joining the mid points of its sides.*

22. Prove that perpendiculars from the corners of a triangle on the corresponding sides of its *first Brocard triangle,* cointersect on its circum-circle. (*Tarry*)

Def′—*The point of cointersection is called the* **Tarry point** *of the original triangle; and the other end of the diameter of the circum-circle through the Tarry point is called the* **Steiner point.**

23. Prove that the join of the circum-centre and Lemoine point is perpendicular to the Simson line of the Tarry point.

24. Prove that the join of the centroid and centre of the T. R. circle, goes through the Tarry point.

25. If G is the centroid of a triangle ABC; and AG, BG, CG cut the circum-circle in α, β, γ respectively; prove that the Lemoine point of the triangle $\alpha\beta\gamma$ is on that diameter which goes through the Tarry and Steiner points. (*Vigarié: Educational Times,* LII, p. 73)

MISCELLANEOUS PROBLEMS.

1. Given the rectangle under two lines, and the difference of their squares, to find the lengths of the lines. (The omitted case of 11, p. 227.)

NOTE—*Let* AB^2 *be the given diff. of sqs., and* ABC *the given rect. Produce* AB *to* X, *so that* $AX . BX = BC^2$. *On* AX *place a semi-*⊙; *and produce* CB *to meet it in* Y. AY, BY *are the reqd. lengths.*

2. Bisect a given triangle by a line through a given point either within or without it.

NOTE—*Let* P *be the pt.,* ABC *the* △. *Draw* AQ *so that* $B\hat{A}Q = C\hat{A}P$, *and* $AQ . AP = \frac{1}{2} AB . AC$: *let segt. on* PQ, *contg.* $B\hat{A}P$, *cut* AB *in* X; XP *is the bisector.*

3. Divide a triangle into parts in a given ratio, by a line parallel to a given direction.

NOTE—*Let* AE, ‖ *to given direc., meet* BC *in* E: *divide* BC *in* D, *in given ratio,* BD *being lesser part: take* BX *mean propl. to* BE, BD: *then* ‖ *thro.* X *to* AE *divides* △ *as reqd.*

4. From a corner of a triangle draw a line to meet the opposite side so as to be a mean proportional between the segments into which it divides that side.

5. Given three collinear points; find a point collinear with them, so that its distance from one may be a mean proportional between its distances from the other two.

6. Given three points, not collinear; find a point whose distances from the three points are proportional to given lengths.

7. Construct a triangle which shall have a given ratio to a given triangle.

8. Construct a triangle of given species, so that the distances of its corners from a given point may be equal to given lengths.

ANALYSIS—*Suppose* ABC *the reqd.* △ *and* P *the given pt.: then, if on* AP *a* △ PAX *is described simr. to* △ CAB, *and* BX *is joined,* △s BAX, CAP *will be simr.,* ∴ CA : BA = CP : BX, ∴ BX *is known, and* △ PBX *can be constructed.*

Exercise 93, p. 85, *gives the construction for the particular case of an equilateral* △.

9. Find a point within a triangle such that its joins to the three corners trisect the triangle.

10. Find O within a triangle ABC so that the circum-circles of AOB, BOC, COA may be equal.

11. Through a given point, between two lines given in position, draw a line so that the segments of it between the point and the lines may have a given ratio.

12. Through a given point, within a given circle, draw a line so that the segments of it between the point and the circumference may have a given ratio.

13. Given a circle, and the positions of two lines; find P on the circle so that PX, PY, drawn parallel to given directions to meet the lines in X, Y, may have a given ratio.

14. From a given point, within a given angle, draw lines to meet the arms of the angle, so as to be in a given ratio and contain a given angle.

15. Through a given point P, within a given angle C, draw a line AB, to make with the arms of the given angle, a triangle ABC of given area.

ANALYSIS—*Let* CQ *be drawn to meet* ⊙ *round* ABC *in* Q, *so that* $A\hat{C}Q = B\hat{C}P$; *then* $\triangle^s$ ACQ, PCB *are simr.*; ∴ CQ . CP = CA . CB, *which is known; and as also* $Q\hat{A}P = P\hat{C}A$, Q *can be found.* Cf. Problem 2.

16. Describe a circle to go through a fixed point, touch a fixed line, and have its centre in another fixed line.

17. Given three points; find a fourth, so that for *every* line through it, if perpendiculars are drawn from the three points on the line, the sum of two of them is equal to the third.

NOTE—*The pt. reqd. is the mean centre of the three pts. when the mults. are each unity.*

18. Given the three altitudes of a triangle, construct it.

NOTE—*Make* △ APQ *so that* AP = p_1, PQ = p_2, AQ = x, *where* $p_3 : p_2 = p_1 : x$. *In* ⊥ *from* A *on* PQ *take* X *so that* AX = p_1. *Thro'* X *draw* ‖ *to* PQ *meeting* AQ *in* B, *and* AP *in* C. ABC *is the* △

19. If X, Y, Z are the points of contact of the in-circle with the sides respectively opposite the corners A, B, C of a triangle; find P, so that—

$$B\hat{P}X = C\hat{P}X,\ C\hat{P}Y = A\hat{P}Y, \text{ and } A\hat{P}Z = B\hat{P}Z.$$

NOTE—*Use* vi. *Addenda* (17).

20. Construct a cyclic quadrilateral, the lengths of whose sides are given.

21. Draw a transversal to a given triangle, so that the segments of it intercepted between the sides (or sides produced) may have given lengths.

22. In a given triangle inscribe another of given species, one of whose sides shall go through a given point.

23. Given the direction of the base of a triangle, and the point at which it is touched by the in-circle; and given also the radius of the in-circle, and the difference of the other sides; find the Locus of the vertex.

24. Construct a triangle when given its vertical angle, the sum of the sides forming that angle, and the difference of the segments of the base made by the foot of the altitude.

25. Find a point in one side of a triangle such that the sum of parallels from it to the other two sides (terminated by them) may be equal to a given length.

26. O is a fixed point, OA a fixed direction; if a circle of fixed radius rolls along OA, and OP is drawn to touch it, and produced to Q, so that OP . PQ = (radius)^2, find the Locus of Q.

27. Given the sum of two sides of a triangle, an angle opposite either of these sides, and the radius of the in-circle; construct the triangle.

28. From a fixed point A *any* line is drawn to meet a fixed line in P; if AQ is drawn so that the angle PAQ, and the rectangle under AP, AQ are of given magnitude, find the Locus of Q.

29. Through fixed points A, B, outside a fixed circle, draw AXP, BYP, so that P, X, Y may be on the circle, and XY parallel to a given direction.

30. ABCD is a quadrilateral which varies subject to the following conditions: the corners A, C are fixed; the species of the triangle BCD is fixed; and the ratio of the rectangles under the opposite sides is fixed: find the Locus of either of the free corners.

NOTE—*Use the construction of* vi. *Addenda* (9), *and the result of* vi. *Addenda* (22).

31. Draw a parallel to one side of a triangle so that of the intercepts between it and that side—

1°, the *sum* = a given length; or
2°, the *diff.* = ,,

32. Given the lengths of the sides of a quadrilateral, and of the join of the mid points of one pair of opposite sides; construct the quadrilateral.

33. Draw the triangle of *minimum* perimeter, which has two corners, one on each of two fixed lines, and the third corner coincident with a fixed point.

When is a solution impossible?

34. Show that the Problem—To inscribe a quadrilateral of *minimum* perimeter in a given quadrilateral is either indeterminate, or impossible.

35. Find the point in one side of a triangle the sum of whose distances from the other two sides is *minimum*.

When is there no *minimum*?

36. Find the point the sum of whose distances from the three sides of a triangle is *minimum.*

37. About a given triangle circumscribe the *maximum* equilateral triangle.

38. With the corners of a triangle as centres describe three circles to touch two and two.

39. Given the base of a triangle, and the length of the line drawn from one end of the base to cut the opposite side in a given ratio; find the Locus of the vertex.

40. Describe a square so as to have its four corners on the sides (or sides produced) of a given triangle ABC.

NOTE—AD is $\perp$ *to* BC; DE, *bisecting* $\widehat{ADC}$, *meets* $\parallel$ *to* BC *thro'* A *in* E; BE *cuts* AC *in* X; CE *cuts* BA *produced in* Y: *then* X, Y *are corners of two sq's solving Prob.*

41. About a given quadrilateral circumscribe a quadrilateral of given species.

42. In a given quadrilateral inscribe a quadrilateral of given species.

43. Given two intersecting circles, and a point in the area common to them, draw the line through the point which divides that common area into parts whose difference is *maximum.*

NOTE—*Take* MN *the chd. of* $\odot$ (*centre* A) *which is* $\perp$ *to* AP *at* P; *and let the* $\odot$ *which is the image of* $\odot$ A, *with respect to* MN, *cut the other* $\odot$ *in* X: XP *is the reqd. line.*

44. Through A, one of the points of intersection of two circles, draw the double chord XAY, so that $a \cdot AX + b \cdot AY = c^2$; where a, b, c are given lengths.

45. Describe a circle so that the angles it subtends at three given points may be respectively equal to given angles.

46. Describe a circle so that the tangents to it from three given points may be respectively equal to given lengths.

47. Find the Locus of the point of contact of two variable circles, which touch two fixed circles and touch each other.

48. Find the Locus of the point from which tangents to two fixed circles are in a given ratio.

49. OX, OY are fixed lines at right angles; and P is a fixed point in the bisector of the angle XOY: find a construction to give X so that the line XPY may be of a given length. *(Pappus' Problem.)*

NOTE—*Drop* PM $\perp$ *to* OX, *and* PN $\perp$ *to* OY; *and produce* MP *to* L, *so that* PL = *given length: with centre* N, *and radius* NL *describe a* $\odot$, *meeting* NP *produced in* H, K: *then the* $\odot$s *on* HP, KP *as diams. will (when a solution is possible) by their intersections with* OX, *give four positions of* X.

50. AB is a fixed finite line; AX, BY are perpendiculars to it; and P is a point in it: if X, Y and P vary subject to the condition that AX . BY = AP . BP, find the Locus of the foot of the perpendicular from P on XY.

51. Given the mid points of the sides of a polygon, can it be constructed?

NOTE—*Let* A_1, A_2, &c., A_n *be mid pts. Take any pt.* P_1, *and draw the trial pol.* P_1 P_2 &c. P_{n+1}, *so that* A_1, A_2, &c., *are respective mid pts. of* P_1 P_2, P_2 P_3, &c. *Join* P_{n+1} P_1; *and bisect it in* X_1. *Draw* X_1 X_2, X_2 X_3, &c., *so that* A_1, A_2, &c. *are their respective mid pts. Join* X_n X_1; *then, if* n *is odd, it will go thro.* A_n; *and* X_1 X_2 &c. X_n *is reqd. pol.*

52. Four rods AB, BC, CD, DA of given commensurable lengths, such that AB + CD = BC + DA, are pivoted together at A, B, C, D, so as to be capable of free angular motion in one plane; if AB is fixed, find the Locus of the in-centre of the quadrilateral formed by the rods. (*Malet*)

53. Given two circles of a co-axal system, describe a circle of the same system to—

1°, go through a given point; or

2°, touch a given line; or

3°, touch a given circle; or

4°, cut a given circle orthogonally; or

5°, cut the join of two given points harmonically.

54. Given six concyclic points A, B, C, D, E, F, find a seventh P, concyclic with them, so that the cross-ratios (PABC) and (PDEF) may be equal.

NOTE—*See General Addenda* ii. (6). *There are two solutions, viz. the pts. in which* XY *meets the* ⊙. *See p.* 411.

55. Solve the last Problem when *collinear* is substituted for *concyclic.*

56. Inscribe a triangle in a given circle, so that its sides may pass respectively through three given points. (*Castillon's Problem*)

NOTE—*If* A, B, C *are the given pts.*; B′C′, C′A′, A′B′ *their polars*; L, M, N *the pts. in which* A′A, B′B, C′C *cut* B′C′, C′A′, A′B′; *then the sides of* △ LMN *will cut the* ⊙ *in six pts.*; *and if these are joined alternately, two* △s *solving the Prob. are obtained. For proof use General Addenda* vii. (5), viii. (4), *and Exercise* 7, p. 376.

57. Circumscribe a triangle about a given circle, so that its corners may be respectively on three given lines.

58. If a triangle has one angle fixed in magnitude and position, and its perimeter is given; find the Envelope of its circum-circle. (*Manheim*)

NOTE—*Invert with respect to the Vertex of the fixed* ∧.

59. Find what the result of the last Problem (*Manheim's*) inverts into with respect to the vertex of the fixed angle as centre of inversion.

60. Given base and vertical angle of a triangle, find the Envelope of its N. P. circle.

61. In triangle ABC, draw XY parallel to BC, meeting AB, AC (or these produced) respectively in X, Y, so that $BX^2 + CY^2 = XY^2$.

62. In triangle ABC, draw XY to cut sides AB, AC in X, Y respectively, so that—

1°, $BX = XY = CY$; or, 2°, $BX + CY = XY$, and XY *minimum*.

63. If S, R are the circumcentre and circumradius of an obtuse-angled triangle; and I, r are the incentre and inradius of the same; find a relation between SO, IO, R and r, where O is the orthocentre.

NOTE—*If* ρ *is the rad. of the polar* ⊙, *prove that*

$$SO^2 = R^2 + 2\rho^2, \quad \text{and} \quad IO^2 = 2r^2 + \rho^2.$$

64. Find the point P, in the plane of a triangle ABC, for which

$$l \cdot PA^2 + m \cdot PB^2 + n \cdot PC^2$$

is *maximum*, where l, m, n are numbers proportional to the areas of the triangles PBC, PCA, PAB respectively.

NOTE—*See Theorem* (20) p. 113.

65. Find the Locus of the inverse of a fixed point with respect to a variable circle which cuts two fixed circles orthogonally.

66. Given a circle and two points A, B within it; inscribe a quadrilateral in the circle so that its diagonals cross in A, and that B is the mean centre of its four corners for equal multiples.

67. Given the incentre, the mid-point of the base, and the foot of the altitude; construct the triangle.

68. Invert any three circles into circles which shall have—

1°, equal radii; or, 2°, collinear centres.

69. A, B, C, D are points in order on a circle; P is a point on the circle, and PB, PC cut AD in X, Y; find the position of P for which XY is *maximum*.

70. A variable circle touches two fixed circles; find the Envelope of its chord of intersection with another fixed circle, concentric with one of the other two.

71. In a triangle ABC, given the difference of AB and AC, find the Envelope of the polar of A with respect to a circle centre C, and given radius R.

72. Through a given point draw a line to be cut harmonically by two given intersecting circles.

73. Three sides of a variable triangle go through three fixed collinear points, and two corners move on fixed lines; find the Locus of the third corner.

74. Describe a circle to cut harmonically each of three given arcs of given circles.

75. Describe a polygon of n sides so that when the extremities of the first side are joined to a given point, the extremities of the second side to another given point, and so on round the polygon, the species of each triangle formed shall be given.

76. A variable circle touches two fixed circles with assigned contacts; find the Envelope of the polar of its centre with respect to one of the fixed circles.

77. Find the Locus of the centre of a variable circle passing through a given point and cutting two fixed circles at equal angles.

78. Find the Locus of the centre of a variable circle cutting three fixed circles at equal angles.

79. If ABC is a triangle; P a point within it; XPX′, YPY′, ZPZ′ parallels respectively to BC, CA, AB, and terminated by the other sides; and if

$$PX \cdot PX' + PY \cdot PY' + PZ \cdot PZ'$$

is given, find the Locus of P.

When is the expression *maximum*? What is its *maximum* value?

Def. If A, A′; B, B′; C, C′; &c. are any number of pairs of collinear points, so connected with another point O, collinear with them, that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \&c.,$$

then the pairs A, A′; B, B′; C, C′; &c., are said to form a *system in involution*: O is called the *centre* of the system: the points in each pair are said to be *conjugate* to each other; and if there is a point F, collinear with the points, so that $OF^2 = OA \cdot OA'$, then F is called a *double point* of the system. Evidently at a double point we may consider that a point of the system coalesces with its conjugate: also, as will be easily seen by drawing diagrams of the construction to be given immediately, there is no double point unless the pairs of conjugate points are all on the same side of O; and, if there is one double point F, there is also a second F′ on the other side of O, such that

$$OF'^2 = OA \cdot OA' = OF^2.$$

The following results are either evident, or can be easily proved.

(1) Each conjugate pair are inverse points with respect to a ⊙, centre O, and radius OF.

(2) Two conjugate points with the double points form a harmonic range.

(3) The cross-ratio of any four points of the system = the cross-ratio of their conjugates. This is sometimes taken as the definition of an involution range; and from it follows that the cross-ratio of a double point with three points of the system = the cross-ratio of the same double point with the three conjugates.

Practically an involution range is generally limited to six points.

To construct for the centre and double points—Draw any ⊙ through a conjugate pair (say A, A′) and another ⊙ through B, B′ and any point P on the first ⊙, cutting it again in Q : QP meets the line of points in the centre O; and the double points F, F′, if they exist, are on opposite sides of O, and given by

$$OF^2 = OF'^2 = OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = \&c.$$

for clearly C′ must be on the ⊙ through C, P, Q.

From the foregoing it is seen that two pairs of points completely determine an involution; and that O is on the radical axis of the ⊙s through P, Q and each conjugate pair.

By joining a range of points in involution to another point, *not* collinear with them, what is called a *pencil in involution* is formed whose focus is this outside point; the rays going through conjugate points are called *conjugate rays*; and the join of the focus of the pencil to the double points of the range are called the *double rays* of the pencil. The points in which a ⊙ through the focus of the pencil crosses conjugate rays are called *conjugate concyclic points*; and the double rays cut this ⊙ in the double points of the system of conjugate *concyclic* points.

To construct the double points of a system of pairs of conjugate concyclic points; taking the test of such a double point to be that the cross-ratio of it with three of the points = the cross-ratio of it with the three conjugates.

Let A, A′; B, B′; C, C′; be three pairs of conjugate concyclic points; and let the two non-conjugate rays AB′, A′B cross in X; the rays BC′, B′C in Y; and the rays AC′, A′C in Z. Then X, Y, Z are collinear. [See *Theorem* (6) on p. 334, where A, D; E, B; C, F; respectively correspond to A, A′; B, B′; C, C′.]

Let YZX meet the ⊙ in Ω; and let AA′ cross XΩ in V. Then

$$A'(\Omega ABC) = A'(\Omega VXZ) = A(\Omega VXZ) = A(\Omega A'B'C').$$

$$\therefore \quad (\Omega ABC) = (\Omega A'B'C');$$

i.e. Ω is a double point of the system.

Simrly. Ω′, where XZY meets ⊙, is the other double point.

The following are fundamental theorems in involution which the Student should prove.

(1) Any line which crosses the sides and diagonals of any quadrilateral is cut in six points in involution.

(2) Any line which crosses a circle and the sides of an inscribed quadrilateral is cut in six points in involution.

(3) The joins of any point to the six corners of a complete quadrilateral form a pencil in involution.

(4) If three chords of a circle concur, the six joins of any point on the circumference to the ends of the chords form a pencil in involution.

By means of double points many Problems, otherwise unmanageable, may be readily solved. The following is a specimen of the mode of operation; and the five Problems succeeding it can be done similarly; so also can Problem 96 on page 423.

80. To inscribe a triangle in a given triangle ABC so that each side may go through one of the given points P, Q, R.

NOTE—*Take any pt.* X_1 *in* BC : *let* $X_1 R$ *cut* CA *in* Y_1 ; *and let* $X_1 Q$, $Y_1 P$ *meet in* Z_1. *If* Z_1 *is on* AB *the problem is solved. If not let* $Y_1 Z_1$ *cross* AB *in* a_1, *and let* $X_1 Q$ *cross* AB *in* β_1. *Construct simrly. for* X_2, X_3. *Let* Z *be the double point of the range* $a_1 a_2 a_3 \beta_1 \beta_2 \beta_3$. *Let* ZP *meet* AC *in* Y, ZQ *meet* BC *in* X, *and* YR *meet* BC *in* X′. *Then*

$$(X X_1 X_2 X_3) = (Z \beta_1 \beta_2 \beta_3) = (Z a_1 a_2 a_3) = (Y Y_1 Y_2 Y_3) = (X' X_1 X_2 X_3).$$

∴ X *and* X′ *are the same point*, *and* XYZ *is inscribed as required.*

81. Given an angle and a point P within it; draw XY to meet the arms of the angle in X, Y, so that XY is parallel to a given direction, and that the angle XPY equals a given angle.

82. Through a given point draw a line crossing four given lines, so that the points of crossing may have a given cross-ratio.

83. On a given line find a point such that its joins with four given points may form a pencil whose cross-ratio is given.

84. Construct a harmonic pencil whose rays pass through four given points, and the two middle ones of which shall contain a given angle.

85. In a given triangle ABC draw AQ to meet BC in Q, so that if circles are inscribed in triangles AQB, AQC, the second external common tangent to these circles may meet BC produced in a given point P.

MISCELLANEOUS EXERCISES.

NOTE—*There is no particular arrangement, either of difficulty* or *otherwise, in the following.*

1. If the bisectors (terminated by the opposite sides) of two angles of a triangle are equal, prove that the sides opposite these angles are equal.

NOTE—*If* BX, CY *are the equal bisectors, complete* $\square$ XCYZ, *and join* BZ; *assume that* $\widehat{ABC} > \widehat{ACB}$; *thence* XC (or YZ) $>$ YB; *and* $\therefore \widehat{YBZ} > \widehat{YZB}$; *but* $\widehat{XBZ} = \widehat{XZB}$; $\therefore \widehat{XBY} < \widehat{XZY}$ (or $\widehat{YCX}$) *contrary to the assumption.*

2. In *Castillon's Problem* (p. 382, Ex. 56) find what conditions in the *data* make the solution indeterminate.

3. Given in a triangle (with the notation of p. 211) the angle A, and $s - a$, find the Envelope of the circum-circle. (*Mankeim*)

4. Given the base of a triangle and the radius of its circum-circle, find the Locus of its in-centre.

5. Prove that the sum of the squares on the twelve lines from the corners of a triangle to the points of contact of its circles of contact with the corresponding opposite sides, is equal to five times the sum of the squares on the sides of the triangle.

6. Prove that the sum of the squares on the tangents from the centres of the four circles of contact of a triangle to *any* circle through the circum-centre, is equal to three times the square on the diameter of the circum-circle.

7. If a, b, c, d are the successive sides of an ordinary quadrilateral; show that, if a circle can be inscribed in it, the process, given in *Note* (1) p. 322, to form it into a *cross-quadrilateral*, fails; and that, in any case, if the process gives a, c, as diagonals, it will not give b, d, as diagonals; and *vice versâ*.

8. Show that a common tangent to two circles subtends a right angle at either limiting point.

9. If A, B are inverse points, show that—

1°, for every point P on the circle of inversion, PA : PB is constant :

2°, if A is inside the circle, the segments of any chord through A subtend equal angles at B ;

3°, if B is outside the circle, the segments of any chord through B subtend supplementary angles at A.

10. Show that any two circles and their inverses are touched by four circles, each of which cuts the circle of inversion orthogonally.

11. If ABC is a triangle, and points P, Q are taken in AB, AC, respectively, such that $BP \,.\, BA + CQ \,.\, CA = CB^2$; show that the Locus of the intersection of BQ, CP, is a circle.

12. If TP, TQ are tangents to a circle; Pp, Qq a pair of parallel chords; and Tt parallel to them, cutting PQ in t; prove that

$$Pp : Qq = Pt : Qt.$$

13. If ABCD is any parallelogram, and P a point within it at which opposite sides subtend supplementary angles; show that the circles PAB, PBC, PCD, PDA are all equal.

14. If on sides AB, AC, of a triangle, isosceles right-angled triangles AEC, AFB are described, both either externally or internally; and if D is the mid point of BC; prove that DEF is an isosceles right-angled triangle.

15. In the sides of a triangle ABC, respectively opposite A, B, C, points D, E, F are taken, so that

$$AF : FB = BD : DC = CE : EA;$$

show that AD, BE, CF cut each other in the same proportion.

16. If ABCD is a cyclic quadrilateral, and E, F are points in CB, CD, such that the angles DAE, BAF are right; prove that EF goes through the centre of the circle round the quadrilateral.

17. ABCD is a cyclic quadrilateral: BA, CD meet in P; and AD, BC meet in Q; if X, Y, Z are the respective mid points of BD, AC, PQ, prove that—

$$AC : BD = 2YZ : PQ = PQ : 2XZ.$$

(*Carr: E. T.* xlvi)

18. If APQ is the tangent at a fixed point A, on a fixed circle, and AP . AQ is constant; prove that the Locus of the intersections of the second tangents from P, Q, is a straight line parallel to APQ.

19. If ABCD is any quadrilateral, and P a point within it at which opposite sides subtend supplementary angles; then, if EF is the third diagonal, prove that the angles APC, BPD, EPF have common bisectors.

20. If A, B are fixed points, in a fixed tangent, to a fixed circle; and X, Y any harmonic conjugates to A, B; find the Locus of the intersection of tangents from X, Y to the circle.

21. Given a triangle, an area, and a ratio; draw a transversal to cut off a triangle equal to the area, and have its segments in the given ratio.

22. S, S′ are inverse points with respect to a circle; CA the radius through S; H the mid point of SS′; if P is any point on the *pedal* of the circle with regard to S, prove that

$$a.r \pm b.r' = \tfrac{1}{2}(a^2 \sim b^2),$$

where $SP = r$, $HP = r'$, $CA = a$, $CS = b$.

NOTE—*The pedal of a* ⊙, *with respect to a pt., is the Locus of the foot of the* ⊥ *from the pt. on any tang. The form of the result shows that this pedal is what is called a focal curve, of the kind whose type is* $r \pm \mu r' = a$; *where* r, r' *are the radii vectores from the foci* S, H; *and* μ, a *are const.: such curves are known as 'Cartesian Ovals.' Taking the case when* S *is within the* ⊙; *if* PN *is* ⊥ *to* SH, *the result comes by eliminating* SH, SN *between*

$r'^2 = r^2 + SH^2 - 2SH.SN$, $2b.SH = a^2 - b^2$, *and* $b.SN = r(a - r)$.

23. If X, Y, Z are points in sides of triangle ABC, respectively opposite A, B, C such that

$$BX : XC = CY : YA = AZ : BZ = \lambda : \mu,$$

prove that $\triangle XYZ : \triangle ABC = \lambda^2 - \lambda\mu + \mu^2 : (\lambda + \mu)^2$.

24. If I_n, C_n are the areas of the regular in- and circum-polygons of *n* sides with respect to the same circle, prove that

$$I_n : I_{2n} = I_{2n} : C_n,$$

$$\text{and} \quad C_n : C_{2n} = C_n + I_{2n} : 2I_{2n}.$$

25. Show that of two regular isoperimetrical polygons, the *maximum* is that which has the greater number of sides.

NOTE—*Hence may be deduced that the* ⊙ *is the maximum area of given perimeter.*

26. Two sides of a given triangle touch two fixed circles, find the Envelope of the third side.

27. If two sides of a given polygon touch two fixed circles, show that all the sides touch fixed circles.

28. (1) Through fixed points A, B, a variable circle is drawn, cutting a fixed circle in X, Y; and XY, AB meet in T: if a variable line TgG is drawn, meeting the fixed circle in g, G, prove that

$$AG.GB : Ag.gB = TG : Tg.$$

(2) Two points A, B, and a circle (centre C) are given: find Z in CA, and V in CB; so that ZT, VT being respectively perpendicular to CA, CB; and TGg a variable line cutting the circle in G, g; then

$$AG.GB : Ag.gB = TG : Tg. \qquad \textit{(Ivory)}$$

29. A, B are inverse points with respect to a circle whose centre is C; P is any point on the circle; BN is perpendicular to the tangent at P; DCE is the diameter perpendicular to CA; PM is the perpendicular on CA (or AC produced) and meets AE in Q: prove that MQ, BN are equal.

30. Show that a variable circle, cutting two given circles at given angles, constantly touches two fixed circles, and cuts a third orthogonally.

31. If a circle constantly touches two fixed circles, show that it cuts any circle co-axal with them at a constant angle.

32. If a circle A cuts a circle B in X, Z; and touches a circle concentric with B in Y; then the arcs XY, YZ are obviously equal: derive a Theorem from this by inverting with respect to a point on the circumference of A.

33. Through each corner of a triangle ABC, parallels are drawn to the opposite sides, forming a new triangle whose sides are YAZ, ZBX, XCY; show that the nine-point circle of ABC touches the nine-point circles of XBC, YCA, ZAB at the mid points of BC, CA, AB, respectively.

34. Through a fixed point, within a fixed angle BAC, draw XPY, so that the perimeter of the triangle AXY is *minimum*.

35. Find the Locus of the centre of a circle cutting two given lines at given angles.

36. A, B, C, D, are concyclic points, in the order named: O_1, O_2, O_3, O_4, are the respective orthocentres of the triangles BCD, ACD, ABD, ABC: prove that—

1°, O_1A, O_2B, O_3C, O_4D, are concurrent; and that,

2°, if the quadrilateral ABCD is turned, in its own plane, round this point of concurrency, through 180°, it will coincide with the quadrilateral $O_1O_2O_3O_4$.

37. If C is the centre of a fixed circle, ACB an angle of fixed magnitude, and APB a tangent to the circle; show that the area ABC will be *minimum* when P is the mid point of the intercepted arc. Hence solve this Problem—

To circumscribe about a given circle a quadrilateral of which the opposite angles shall be of given magnitude, and the area *minimum*.

38. Given a circle and a fixed point A, prove that another fixed point B can be found such that, if a tangent is drawn at a variable point P, AP^2 will vary as the perpendicular from B on this tangent.

NOTE—*Recollect that a tang. is the polar of its own pt. of cont.*, *and use Salmon's Theorem.*

39. If on the sides of any triangle equilateral triangles are described (all externally, or all internally to the triangle) show that the joins of their centres form an equilateral triangle. (*Cf.* pp. 318, 319.)

40. Is it possible to inscribe in a circle a pentagon equiangular to *any* given pentagon?

41. Given two circles, find a point such that tangents from it to the circles may be equal, and the angle between them of given magnitude.

42. Given a quadrilateral; find a point such that, if perpendiculars are dropped from it on the sides, the joins of their feet form a parallelogram.

43. If A is the intersection of a direct and a transverse common tangent to two circles, and AL perpendicular to their line of centres, show (by aid of Ex. 8, p. 413) that L is a limiting point of the circles.

Deduce (using Ex. 9, p. 352) that, if D is the diameter of the circle orthogonal to the three ex-circles of a triangle, r the in-radius, and s the semi-perimeter of the triangle, then $D^2 = r^2 + s^2$. (*E. T. Vol.* LVII. *p.* 87.)

44. In *Pascal's Theorem* (p. 334) if the hexagon varies subject to the limitations that the circle, and the three collinear points of intersection, are fixed, prove that any diagonal goes through a fixed point.

45. If one pair of opposite sides of a quadrilateral inscribed in a fixed circle touch another fixed circle, show that the other pair of opposite sides touch a third fixed circle co-axal with the two former.

Hence prove *Poncelet's Theorem*—If all the sides but one of a variable polygon inscribed in a fixed circle touch fixed circles co-axal with the first, the remaining side touches another fixed circle of the system.

46. Two tangents are drawn to a circle, and two lines dividing the angle between the tangents harmonically; show that the pole of one of these lines lies on the other.

NOTE—*An* $\widehat{AOC}$ *is divided harmonically by* OB, OD, *when,* ABCD *being a transversal,* O (ABCD) *is harmonic.*

47. XYZ is a transversal to a triangle, such that the ratio of XY to YZ is constant; if XY is divided in P, so that the ratio XP to PY is constant, find the Locus of P.

48. In a segment of a circle inscribe the rectangle of *maximum* area.

NOTE—*By General Addenda* i. 3, *if tangs. are drawn to the arc of the segt. at the corners of an inscribed rect., forming with the chd. produced of the segt. a* $\triangle$, *the rect. is max. when then the pts. of cont. are mid points of sides of* $\triangle$. *Now let* X *be pt. where a tang. meets chd.;* P *pt. where* $\perp$ *at* X *to chd. meets rad. thro. pt. of cont.;* M *pt. where* PX *meets* $\parallel$ *to chd. thro.* O, *the centre of* $\odot$*: then* PM . PX = 2 (rad.)^2, *and* PM − PX *is known;* $\therefore$ PM *can be found; and then* $\triangle$ PMO *is known.*

49. Given base, difference of base angles, and rectangle under the sides, construct the triangle.

50. Given a point and an angle, draw through the point a line so that the length of it intercepted between the arms of the angle may be given.

Reduce to the preceding Problem the following—Given a semi-circle and a line; draw a tangent to the semi-circle, so that its intercept between the line and the diameter may be divided at the point of contact in a given ratio.

51. With the notation of p. 212, if U is taken in IE, so that $IU = \frac{1}{4} IE_1$; and V, W are taken similarly; and if α, β, γ are mid points of sides of triangle $E_1 E_2 E_3$; show that the circum-circle of UVW bisects IA, IB, IC, $I\alpha$, $I\beta$, $I\gamma$.
(*J. Griffiths*)

52. Construct a triangle of given species, with its corners on three concentric circles.

If the triangle is equilateral, find the relation between the radii that there may be only one solution.

53. Find a point in a given line so that the sum (or difference) of its distances from two fixed points may be given.

Reduce to the preceding Problem the following—Describe a circle, with its centre on a given diameter of a given circle, to cut that circle orthogonally, and touch another given circle.

54. If P is a given point outside a triangle, and Q a given point in one of its sides; draw PXY to cut the other sides in X, Y, so that the angle XQY may be given.

When will the angle XQY be *maximum*?

55. If a circle touches two sides of a triangle and its circumcircle, prove by inversion that the join of the points of contact with the sides goes through the in-centre when the contact of the circles is internal, and through an ex-centre when external. (E. T. *Vol.* LV. *p.* 107)

56. Draw the *maximum* triangle, of given species, so that each side touches one of three given circles.

57. If a circle rolls (without slipping) on the circumference of a circle of double its radius (the contact being internal) show that each of the points of the lesser circle goes along a diameter of the larger. (*La Hire*)

58. If a circle rolls on a fixed circle of half its radius (the contact being internal) prove that the Envelope of any chord of the rolling circle is a circle, which reduces to a point when the chord is a diameter. (*Wolstenholme*)

59. Given the in- and circum-circles of a triangle, find the Locus of—
1°, its orthocentre; 2°, its three ex-centres.

60. OA, OB are fixed lines; C, D fixed points inside the angle AOB; draw a circle, with O as centre, cutting OA, OB in X, Y, so that, if DX, DY are joined, CX + DY may be *minimum*.

61. In a given circle draw two parallel chords, so that they shall be to each other in a given ratio, and be at a given distance apart.

62. Given three concurrent lines, and a triangle, construct a triangle with its corners on the lines, and identically equal to the given triangle.

63. A variable circle goes through a given point and touches a given circle S; prove that its chord of intersection with another fixed circle, concentric with S, touches a fixed circle.

64. A, B, C, D are four concyclic points: a system of circles is drawn having A, B for limiting points; and a second system having C, D for limiting points: find the Locus of the points of contact of a circle of the first system with a circle of the second system.

65. If the join of two points on two circles subtends a right angle at a limiting point, prove that the Locus of the intersection of tangents at the points is a co-axal circle. (*W. S. McCay*)

66. If ABC is a triangle, and S its circumcentre; find the point P for which the ratio of Σ PA to PS is *minimum*. (*Prof. Purser*)

NOTE.—P *is the inverse of* Fermat's point. *See p.* 326.

67. If two circles are inverted into two others; then if Ot, Ot′ are tangents from the centre of inversion to the second pair, and R the radius of inversion, prove that—

common tangent to 1st pair : common tangent to 2nd pair = $R^2 : Ot \cdot Ot'$

(*Prof. Purser*)

NOTE—*Let* PQ *be common tang. to* 1*st pair;* p, q *inverse, and* p′, q′ *corresponding pts. of* P, Q; pq *produced meets* ⊙ *thro.* p, p′ *in* x, *and* ⊙ *thro.* q, q′ *in* y. *Easy to see that* x, q *are corresponding pts., and also* y, p.

Then (*common tang. to* 1*st pair*)^2 : (*common tang. to* 2*nd pair*)^2

$= PQ^2 : py \cdot qx$ [cf. p. 345, Ex. 10]

$= (PQ : pq)\ (pq : py)\ (PQ : pq)\ (pq : qx)$

$= (OP : Oq)\ (Oq : Oq')\ (OQ : Op)\ (Op : Op')$ [*Simr.* Δ^s *and*

$= (OP : Op')\ (OQ : Oq')$ *Componendo*]

$= (OT : Ot)\ (OT' : Ot')$

$= (R^2 : Ot^2)\ (R^2 : Ot'^2)$.

Whence result follows by converse part of v. 9.

NOTE—*The remaining Exercises are taken from the Geometry Papers set, at the Royal University of Ireland, to candidates for the Mathematical Scholarships given to commencing Students.*

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68. If a chord of a given circle subtend a right angle at a given point, the rectangle contained by the perpendiculars on it, from the given point and from the centre of the given circle, is constant. Also the sum of the squares of perpendiculars on it from two other fixed points (which may be found) is constant.

69. The sum of the squares of lines drawn from the angular points of a regular polygon of n sides to any point in the circumference of its inscribed circle is equal to n times the sum of the squares of the radii of its inscribed and circumscribed circles.

70. If a, b, c denote the sides of a triangle ABC; D, D′ the points where the internal and external bisectors of the angle A meet the opposite side, prove that $DD' = \frac{2\,abc}{b^2 - c^2}$.

And, in the same case, if E, E′, and F, F′, be points similarly determined on the sides CA, AB respectively, prove that

$$\frac{1}{DD'} + \frac{1}{EE'} + \frac{1}{FF'} = 0$$

and
$$\frac{a^2}{DD'} + \frac{b^2}{EE'} + \frac{c^2}{FF'} = 0.$$

NOTE—*It is sometimes difficult, if not impossible, to interpret results like these within the limits of Euclidian Geometry. The Student, in proving them, should in general avoid the use of the word 'multiply'; which is an arithmetic term, and always tacitly assumes that the multiplier is a commensurable number.*

71. If two circles intercept on any secant chords that have a given ratio, the tangents to the given circles at the points of intersection with the secant have a given ratio.

72. If a, b, c, d denote the four sides; and D, D′ the diagonals of a quadrilateral; prove that the sides of the triangle formed by joining the feet of the perpendiculars from any of its angular points on the sides of the triangle formed by the three remaining points, are proportional to the three rectangles ac, bd, DD′.

73. If a variable circle touch two fixed circles in the points A, B; and the line AB be drawn intersecting the fixed circles again in A′, B′; and if DD′ be the common tangent of the two fixed circles; prove that

$$DD'^2 : AB^2 = (R \pm \rho)(R \pm \rho') : R^2,$$

where R is the radius of the variable, and ρ, ρ' of the fixed circles; and the choice of sign depends on the nature of the contact.

74. If four circles are touched by a fifth; and if we denote by $\overline{12}$ the common tangent to the first and second, and use a similar notation for the others; prove that $\overline{12} . \overline{34} + \overline{23} . \overline{41} = \overline{13} . \overline{24}$.

NOTE—*This is Dr. Casey's well-known extension of Ptolemy's Theorem: it may be best proved thus—Invert with respect to a p't on the* 5th ⊙, *and we get* 4 ⊙s *touching a line; apply Euler's Theorem* (p. 104) *to the seg'ts of this line made by the p'ts of contact; and connect these seg'ts with the tang's* $\overline{12}$, &c., by *Purser's Theorem* (p. 389).

75. If three concurrent lines from the angles of a triangle ABC meet the opposite sides in the points A′, B′, C′, the diameter of the circle circumscribed about ABC is equal to AB′ . BC′ . CA′ divided by the area of the triangle A′B′C′.

76. Prove the *minimum* property of *Philo's Line.*

NOTE—*See p.* 426.

77. The base AB, of a given triangle ABC, is cut harmonically in X and Y; show that the circle circumscribing the triangle XCY passes through a second fixed point.

78. Determine a point P such that its three distances from the vertices of a given triangle may bear to one another given ratios. Show that, if two real positions of the point P exist, the line joining them passes through the centre of the circumscribing circle of the given triangle.

79. Draw a line which shall cut off similar segments from three given circles.

How many solutions does the Problem admit of?

80. ABC is a fixed triangle; A′B′C′ is a similar triangle of *opposite sense*, lying inside the first; show that, if a, b, c are the sides of the outer, p_1, p_2, p_3 perpendiculars let fall from A′, B′, C′ on these sides,

$$ap_1 + bp_2 + cp_3 = \text{twice area outer triangle.}$$

Def′—Two similar figures may be said to be of *opposite sense* if, in order to place the sides of one parallel to those of the other, it would be necessary to invert one of their planes. (Same as *inversely similar* p. 378.)

81. Describe a pentagon of *maximum* area, such that four of its sides shall have given and equal lengths, and the angles at the extremities of the fifth side shall be right angles.

82. From a variable point P perpendiculars are let fall on the sides of a given triangle: find the Locus of P, if the area of the triangle formed by joining the feet of the perpendiculars is constant.

83. Two variable lines, OA and OB, making with one another a given angle, are drawn from a given point O to meet a given line: circles are drawn touching the given lines at A and B, and each passing through O; find the Locus of their other point of intersection.

NOTE—*Invert with respect to* O.

84. A given circle is touched at two fixed points by two variable circles, which also touch one another; find the Locus of their point of contact.

85. On a given straight line find a point whose distance from a fixed point is equal to its distance from a fixed line. State the conditions as to the *data* which would render the Problem impossible.

86. Prove that all circles touching two given circles are cut orthogonally by one or other of the two circles which pass through the intersections of the given ones, and bisect the angles between them.

NOTE—*Invert with respect to one of the p'ts of intersection of the given* ⊙s.

87. Three equal circles ABB′A′, ACC′A′, CBB′C′, intersect in any manner; but so that B, B′, the intersections of the first and third, lie within the second: prove that arc AB + arc BC − arc AC = arc A′B′ + arc B′C′ − arc A′C′.

88. A point P moves so that $m \cdot AP - \frac{1}{m} \cdot BP = AB$; A and B being two fixed points: show that a point X can be found in AB, such that the ratio of XP to AP + BP shall be constant.

NOTE—X *divides* AB *so that* $AX : BX = 1 : m^2$.

89. Two given circles turn round two fixed points A, B, on their circumferences, in such a manner that one of their intersections describes the line AB: find the Locus of their second intersection.

90. Through the vertices of a triangle ABC lines are drawn to a point O; if any triangle have its sides parallel to OA, OB, OC; and if through each of its vertices a parallel to the corresponding sides of the original triangle be drawn; prove that these three parallels meet in a point.

91. Being given the sides of the squares inscribed in a right-angled triangle, construct the triangle.

92. Three sides a, b, c, of a quadrilateral are given; and its area is a *maximum*; prove that the remaining side x satisfies the equation

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

93. One of the vertical angles of a triangle is fixed in magnitude and position, and the circumscribing circle passes through another fixed point A; find the Locus of the foot of the perpendicular let fall from A on the variable side of the triangle.

94. If through a fixed point O a variable right line be drawn, cutting any number of given circles in pairs of points (A, A′) (B, B′) &c., find the Locus of the point X on the line such that, if l, m, n are given constants,

$$\frac{1}{OX} = l\left(\frac{1}{OA} + \frac{1}{OA'}\right) + m\left(\frac{1}{OB} + \frac{1}{OB'}\right) + \&c.$$

95. If four circles be such that each cuts the other orthogonally; prove that the centres of three of them are the vertices of a triangle self-reciprocal with respect to the fourth.

96. Through a given point A draw a right line, such that the right line joining its poles, with respect to two given circles, shall pass through another fixed point.

97. If A′, B′, C′ be the feet of the perpendiculars from the vertices A, B, C of a triangle on the opposite sides; prove, if A″, B″, C″ be the centres of the circumscribed circles of the triangles AB′C′, BC′A′, CA′B′, respectively; and α, β, γ the centres of their inscribed circles; $A''\alpha$, $B''\beta$, $C''\gamma$ are concurrent.

NOTE—*They concur where* N. P. *and in-*$\odot$s *of* $\triangle$ABC *touch.*

98. Inscribe a triangle in a circle so that two sides shall pass through two given points, and that the third side shall be parallel to a given line.

99. If from any point of the *Pascal's Line* of a hexagon, inscribed in a circle, perpendiculars be let fall on the sides; prove that the product of the perpendiculars on three alternate sides is equal to the product of the perpendiculars on the three remaining sides.

NOTE—*If we call the* $\perp$s $p_1, p_2, p_3, p_4, p_5, p_6$, *it will be more geometrical to put the result in the form* $(p_1 : p_2)(p_3 : p_4)(p_5 : p_6) = 1$.

100. If a circle be inverted from any arbitrary point, prove that the inverses of the angular points of any inscribed square will form a harmonic system of points.

101. Prove that the difference of the squares of the two interior diagonals of a cyclic quadrilateral is to twice their rectangle, as the distance between their middle points is to the third diagonal.

NOTE—*Use Carr's Theorem*, p. 414.

102. Prove that upon a given line can be described six triangles equiangular to a given triangle; and that their six vertices are concyclic. (*Neuberg*)

103. If two triangles are given, in all but position, place them so that the vertices of one may be the poles of the sides of the other, with respect to a circle.

104. ABC is a given triangle; L, M, N its three ex-centres; show that the Problem—To draw a triangle whose vertices shall lie on the sides of ABC, and whose corresponding sides shall pass respectively through L, M, N—is indeterminate.

105. A quadrilateral is inscribed in one circle and circumscribed to another; show that its diagonals meet in a limiting point of the two circles.

106. The poles of the radical axis of two circles are harmonic conjugates with respect to their centres of similitude.

107. Given in position, but not in length, one pair of opposite sides of a cyclic quadrilateral, and the intersection of its diagonals; find the Locus of the centre of its circumscribing circle.

108. A, B, C, D are four fixed points on a circle; X, Y two variable points on the same: show that the line joining the intersection of the chords AY and CX to that of the chords BY and DX, passes through a fixed point.

109. Two maps of the same country, on different scales, are placed anywhere on a table; one of the maps, which is drawn on transparent paper, being turned upside-down upon its face: find the point on the table which represents the same place, to whichever map it is considered to belong.

110. Through a given point draw a right line so as to be cut harmonically by two given intersecting circles.

111. If CA, CB are tangents to a circle, show that, if any third tangent cuts them in P, Q, the rectangle AP . BQ is to the rectangle CP . CQ in a constant ratio.

112. If A, B are inverse points with regard to a circle, and any point P be taken on the circle, and AP, BP intersect the circle again in X, Y; given AB = c, AP = r, BP = r′, find the distances of X, Y from A and B.

113. The vertical angle of an isosceles triangle is 120°; find the Locus of a point P, if the sum of the squares of its perpendicular distances from the two sides is equal to the square of its distance from the base.

NOTE—*The Locus is the polar* ⊙ *of the* △.

114. From what point can the three sides of a triangle be inverted into three equal circles?

NOTE—*The in-centre, and the ex-centres.*

115. Through a fixed point O, within a circle, any two chords AB, CD are drawn: find the Locus of the intersection of AC and BD.

116. Given a triangle ABC, show that a point P may be found, such that every circle through C and P shall cut the sides AC, BC in points R and S, such that AR : BS is a constant ratio.

117. Given three points A, B, C, draw a line through C, such that the sum, or difference, of the squares of the perpendiculars on it from A, B shall be given.

118. If A, B, C are three given points on a line (AB > BC) and *any* point P be taken, such that AP, BP, CP are in descending geometrical progression; show that all positions of P lie within the circle passing through B, and having A, C as inverse points.

119. Given three points A, B, C; find a point P, such that PA + PB shall be equal to one given line, and PA + PC to another.

120. AB, a chord of a fixed circle, subtends a right angle at a fixed point O: circles are described through O, touching the fixed circle at A and B respectively: prove that the Locus of their other intersection is a circle.

121. From a given point P are drawn three lines, meeting a given line in the points A, B, C, respectively: prove that, if the radii of the circles inscribed in the triangles PAB, PBC, is given, then the radius of the circle inscribed in PAC is given.

122. If the sum of two opposite angles of a quadrilateral is three right angles; prove that the sum of the squares of the rectangles contained by the pairs of opposite sides is equal to the square of the rectangle contained by the diagonals.

NOTE—*The "square of a rectangle" is a phrase with no* Geometrical *meaning.*

123. Construct a quadrilateral, given the four sides and the area.

NOTE—*Lemma: given hypots.* of 2 *right-angled* $\triangle^s$; *also given sum of one pair of sides, and diff. other pair of sides; the* $\triangle^s$ *can be constructed thus:* AB = *sum*, BC = *diff.*, ABC *being rt.* $\wedge$; *inflect* AD, CD, *the hypots; and drop* DE $\perp$ *on* AB, CF $\perp$ *on* DE: *then* AED, CFD *are the* $\triangle^s$.

ANALYSIS—*Let* ABCD *be quad.*: *produce* BC *to* G, *so that* AD : AB = BC : BG; *and drop* GH, CF $\perp^s$ *on* AB, *and* CE *on* AD. *Then, easily,*

$$2AD\,(DE - BH) = 2AD\,.\,DE - 2AB\,.\,BF = AB^2 + BC^2 - AD^2 - CD^2.$$

$$\therefore\ DE - BH \textit{ known.}$$

$$\textit{Also } 2\textit{ area} = AD\,.\,CE + AB\,.\,CF = AD\,(CE + GH).$$

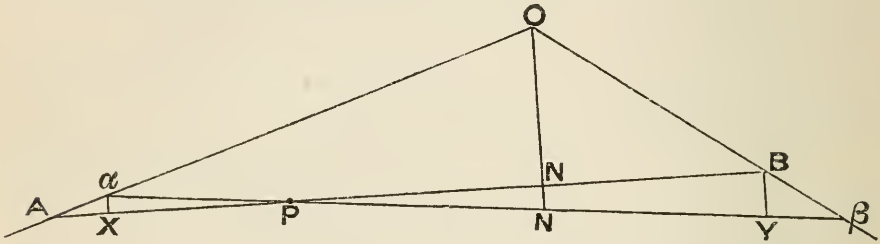
$$\therefore\ CE + GH \textit{ known.}$$

$$\therefore\ \textit{Prob. reduced to Lemma.}$$

The following will be found a more simple arrangement, than that usually given, of the proof of the property stated in the Note on p. 284.

Ax'—A continuously varying geometrical magnitude is in a position of *maximum* or *minimum* value when for indefinitely close consecutive positions it has the same value (*Fermat's Principle*).

THEOREM—*If* AOB *is any angle,* P *any point within it,* APB *a line terminated by the arms of the angle, and* ON *perpendicular to* APB ; *then, for the position when* APB *is* minimum, PA *and* BN *are equal.*



There is obviously no *max'* ; and ∴, when consecutive positions APB, $\alpha P\beta$, are equal, the line (by the above ax') is *min'*.

Let ⊙, centre P, rad' $P\alpha$, cut PA in X ; and ⊙, centre P, rad' PB, cut $P\beta$ in Y.

$$\text{Then } AX = AB - PX - PB = \alpha\beta - P\alpha - PY = \beta Y.$$

Now, ultimately, when AB, $\alpha\beta$ are indefinitely close together, αX, BY are respectively ⊥ to PA, $P\beta$; and ON is ⊥ either to AB or $\alpha\beta$: also we may then put BN for $N\beta$, and PA for $P\alpha$.

∴, ultimately, by sim'r Δ^s, we have

$$AN : NO = AX : X\alpha,$$

$$\text{and } ON : N\beta = BY : Y\beta.$$

$$\therefore, \textit{ex æquali}, \quad AN : N\beta = BY : X\alpha$$

$$= BP : P\alpha$$

$$\text{or } AN : BN = PB : PA.$$

$$\therefore, \textit{componendo}, \quad AB : BN = AB : AP.$$

$$\therefore \quad BN = AP.$$

Note—This *min'* position is called **Philo's Line** : cf. p. 284.

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THE END.

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NOTE.—*The following refer to the new section (originally published separately) on the Lemoine and Brocard Geometry.*

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